Divisors on moduli spaces of level curves

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Dipl.-Math. Gregor Bruns

Präsidentin der Humboldt-Universität zu Berlin
Prof. Dr.-Ing. Dr. Sabine Kunst

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät
Prof. Dr. Elmar Kulke

Gutachter:
1. Prof. Dr. Gavril Farkas
2. Prof. Dr. Carel Faber
3. Prof. Dr. Alessandro Verra

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Dedicated to JBA.

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Abstract

In this thesis we investigate three questions. Two are about divisors on moduli spaces of level curves, and about the consequences for the birational geometry of these spaces. The third asks about the stability properties of normal bundles of canonical curves.

The first question, to be studied in Chapter 2, is about the Kodaira dimension of the moduli space $\mathcal{R}_{15,2}$ of Prym varieties of genus 15. We study a new divisor on this space and calculate its class in terms of the standard basis of the Picard group. This allows us to conclude that $\mathcal{R}_{15,2}$ is of general type.

Continuing the study of level curves in Chapter 3, we investigate, for every ℓ, theta divisors on $\mathcal{R}_{6,\ell}$ and $\mathcal{R}_{8,\ell}$ defined in terms of the *Mukai bundle* of genus 6 and 8 curves, respectively. These bundles provide canonical embeddings of our curves in Grassmann varieties and describe fundamental aspects of the geometry of curves of these genera. Using the class of the divisor for $g = 8$ and $\ell = 3$, we are able to prove that $\mathcal{R}_{8,3}$ is of general type as well.

Finally, in Chapter 4 we study the stability of the normal bundle of canonical genus 8 curves and prove that on a general curve the bundle is stable. For canonical genus 9 curves we prove stability at least with respect to subbundles of low ranks. We also provide some more evidence for the conjecture of M. Aprodu, G. Farkas, and A. Ortega that a a general canonical curve of every genus $g \geqslant 7$ has stable normal bundle.

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CHAPTER 1

Background material

1.1 Moduli spaces of stable curves

Algebraic curves have been at the center of interest of (algebraic) geometry since its very beginning, but at first only individual curves and their properties were studied. It was Riemann who in the 19th century envisioned a space whose points would correspond to isomorphism classes of smooth curves of a fixed genus g. He was able to prove ([Rie57]) that such a space would have dimension $3g-3$, i.e., the isomorphism class of a curve would depend on $3g-3$ parameters, which he called *moduli*. The calculation was done by exhibiting every smooth curve of genus g as a ramified cover of the Riemann sphere $\mathbb{P}^1$.

It should however take geometers almost a century to rigorously prove the existence of such a parameter space. In the meantime, many calculations and constructions were done where its existence was implicitly assumed. The first construction of $\mathcal{M}_g$, the symbol coined for the moduli space of smooth genus g curves, as an algebraic variety was put forward by Mumford in 1965 ([MF82]).

Convention 1.1. We will always work over the complex numbers $\mathbb{C}$ and our curves will be connected and of genus $g \geqslant 2$.

The space $\mathcal{M}_g$ constructed by Mumford is indeed an irreducible algebraic variety of dimension $3g-3$ ([DM69]) whose closed points correspond to isomorphism classes of smooth genus g curves. Not everything was achieved with this construction, though: $\mathcal{M}_g$ does not quite have the properties of what we call a *fine moduli space*. We would like that morphisms $\varphi_B\colon B \to \mathcal{M}_g$ from any variety B correspond to families $\mathcal{X} \to B$ of smooth curves of genus g over B. This would also imply the existence of a *universal family* $\mathcal{C} \to \mathcal{M}_g$ such that $\mathcal{X} \to B$ arises as the pullback along φ_B:

$$\begin{tikzcd} \mathcal{X} \arrow[r, dashed] \arrow[d] & \mathcal{C} \arrow[d, dashed] \\ B \arrow[r] & \mathcal{M}_g \end{tikzcd}$$

However, due to the existence of curves with non-trivial automorphisms, this correspondence between maps and families turns out not to hold true. On the other hand, given a family $\mathcal{X} \to B$ we do in fact get a morphism $B \to \mathcal{M}_g$. This, together with the bijection between closed points of $\mathcal{M}_g$ and the isomorphism classes of smooth genus g curves, makes $\mathcal{M}_g$ into a *coarse moduli space*.

To remedy the defect of $\mathcal{M}_g$ it is often convenient to use the language of stacks. Indeed, the stack M_g of families of smooth curves of genus g satisfies, basically by definition, the properties of a fine moduli space. Its coarsening is precisely $\mathcal{M}_g$. While M_g is a smooth Deligne–Mumford stack, the coarse space has singularities at points corresponding to curves with nontrivial automorphisms.

Since $\mathcal{M}_g$ is not compact, but quasi-projective, it is natural to ask for a compactification. The first idea of embedding $\mathcal{M}_g$ in projective space by a very ample line bundle and taking the compactification there does not lead to satisfactory results: the points on the resulting boundary do not naturally correspond to curves that fit into families with the smooth curves $[C] \in \mathcal{M}_g$. It were P. Deligne and D. Mumford in [DM69] who introduced the notion of a stable curve in order to obtain a better compactification of $\mathcal{M}_g$.

Definition 1.2. A *stable curve* of genus g is a connected nodal curve C with $h^1(C, \mathcal{O}_C) = g$ and ω_C ample.

The condition of ω_C to be ample is equivalent to $\mathrm{Aut}(C)$ being a finite group. More concretely this means that every smooth rational component of C meets the rest of the curve in at least 3 points.

The compactification that results from taking all stable curves into account is denoted by $\overline{\mathcal{M}}_g$ and is called the Deligne–Mumford compactification. It is indeed modular in the sense that boundary points correspond precisely to singular stable curves, and these arise as degenerations of smooth curves in families. Indeed, the properness of $\overline{\mathcal{M}}_g$ says that every flat family of smooth curves of genus g over a punctured disc $B^* \subset B$ can (after possibly a finite base change) be uniquely extended to a flat family over B such that the special fiber is a stable curve of genus g. This fact is also know as the *stable reduction theorem*.

1.1.1 Boundary divisors

With the Deligne–Mumford compactification available, we can begin the study of its divisor theory. In order to get a basic understanding, we will first describe the boundary $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$. It consists of several irreducible components denoted $\Delta_0, \ldots, \Delta_{\lfloor g/2 \rfloor}$. Their general points correspond to stable curves of a certain topological type.

The general curve $[C] \in \Delta_0$ is an irreducible one-nodal curve:

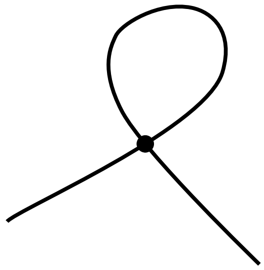

Figure 1.1: An irreducible one-nodal curve.

We can obtain such a curve by taking any $[C'] \in \mathcal{M}_{g-1}$ and identifying two points $p, q \in C'$. On the other hand, the general curve $[C] \in \Delta_i$ for $i \geqslant 1$ has one smooth component of genus i and one of genus $g - i$, meeting at a node:

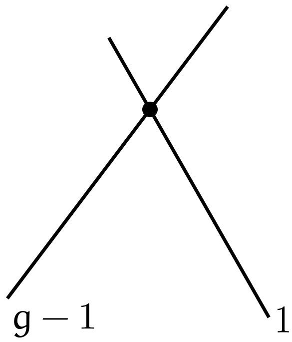

Figure 1.2: A curve of compact type in Δ_1.

Because no component is preferred over the other, e.g., by some marking, we have $\Delta_i = \Delta_{g-i}$ and hence we can restrict to $i \leqslant \lfloor g/2 \rfloor$.

The intersection of boundary components is easy to understand. For instance, the general points of the intersection $\Delta_0 \cap \Delta_1$ have precisely one irreducible node:

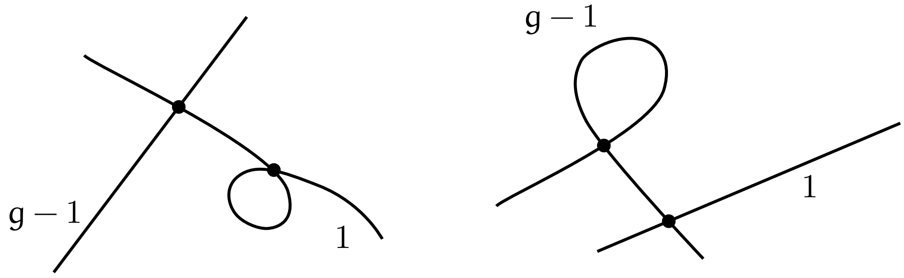

Figure 1.3: Typical curves in $\Delta_0 \cap \Delta_1$.

1.2 Moduli spaces of twisted level curves

Almost as classical as considering genus g curves is to consider curves together with some additional data, e.g., points, theta characteristics or torsion line bundles. Well-studied and of particular interest are moduli spaces of Prym varieties.

A *Prym variety* is an abelian variety associated to an unramified double cover $\pi\colon C' \to C$ of curves in the following way. If g is the genus of C, then we can produce an abelian variety $\mathrm{Pr}(C, \pi)$ of dimension $g-1$ from the cover π by considering the *Norm map*

$$\mathrm{Nm}_\pi\colon \mathrm{Pic}^{2g-2}(C') \to \mathrm{Pic}^{2g-2}(C), \quad \mathcal{O}_{C'}(D) \mapsto \mathcal{O}_C(\pi_* D)$$

and then letting

$$\mathrm{Pr}(C,\pi) = \mathrm{Nm}_\pi^{-1}(K_C)^+ = \left\{ L \in \mathrm{Nm}_\pi^{-1}(K_C) \;\middle|\; h^0(C, L) \equiv 0 \pmod 2 \right\}$$

If we let $\Theta = W_{2g-2}(C')$ be the Riemann theta divisor of C' then we get a relation

$$\Theta \cdot \mathrm{Pr}(C,\pi) = 2\,\Xi_C$$

where Ξ_C turns out to be a principal polarization of $\mathrm{Pr}(C,\pi)$.

More generally, we can associate a *Prym–Tyurin variety* to an endomorphism γ of the Jacobian $\mathrm{Jac}(C')$ of a curve C' by letting $P = \mathrm{im}(1-\gamma)$. If γ satisfies a quadratic equation, then P is principally polarized.

We now fix an integer ℓ and consider the moduli space $\mathcal{R}_{g,\ell}$ parametrizing isomorphism classes of pairs $[C, \eta]$, where $[C] \in \mathcal{M}_g$ is a smooth genus g curve and η is a point of order ℓ in the Jacobian of C, i.e., a line bundle of degree 0 on C with $\eta^{\otimes \ell} \cong \mathcal{O}_C$ and $\eta^{\otimes k} \neq \mathcal{O}_C$ for all $0 \leqslant k < \ell$. In what follows we will in fact only be concerned with prime numbers ℓ. This simplifies matters considerably, but the results carry over to composite ℓ almost word by word (see the discussion in [CEFS13]).

With this definition, $\mathcal{R}_{g,2}$ is the moduli space of Prym varieties discussed above. To a pair $[C, \eta] \in \mathcal{R}_{g,2}$ we can associate an unramified cover $\pi\colon C' \to C$ of degree 2 by letting $C' = \mathcal{S}pec(\mathcal{O}_C \oplus \eta)$. Conversely, we can retrieve the pair $[C, \eta]$ from such a cover by letting $\eta = \det(\pi_* \mathcal{O}_{C'})$.

Hence we get a morphism, called the *Prym map*, from $\mathcal{R}_{g,2}$ to the moduli space $\mathcal{A}_{g-1}$ of principally polarized abelian varieties of dimension $g-1$. Prym varieties play an important role in the study of the moduli spaces $\mathcal{A}_g$ since a general abelian variety of dimension at most 5 is a Prym. On the other hand, recall that the general abelian variety of dimension at least 4 is not the Jacobian of a curve. Hence Prym varieties make the study of abelian varieties amenable to techniques from curve theory in a larger range than by just studying Jacobians.

By a similar procedure, one can assign a cyclic unramified cover $C' \to C$ of degree ℓ to a pair $[C, \eta] \in \mathcal{R}_{g,\ell}$ for $\ell \geqslant 3$ as well. However, this process is only reversible if we consider such covers together with a generator of their Galois group.

Several constructions to compactify $\mathcal{R}_{g,\ell}$ have been put forward. The first one was A. Beauville with his theory of admissible covers for $\mathcal{R}_{g,2}$ ([Bea77] and [ACV03]) which extends the modular description of points in $\mathcal{R}_{g,2}$ as étale double covers $C' \to C$ to stable curves C. Later, M. Bernstein in her PhD thesis considered the normalization of $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{R}_{g,\ell}$. Points of the ensuing compactification $\overline{\mathcal{R}}_{g,\ell}$ correspond to stable curves with torsion line bundles on each component, at least for curves of compact type. She then realized ([Ber99, Theorem 2.6.4]) that over irreducible nodes we need to additionally consider ℓ-th roots of line bundles of the form $\mathcal{O}_{\widetilde{C}}(ap + (\ell - a)q)$. Here $\widetilde{C} \to C$ is the normalization and $p, q \in \widetilde{C}$ are the two points lying over the node. For a more precise description see section 1.2.1 below.

Later on, D. Abramovich, A. Corti and A. Vistoli ([ACV03]) used stacky curves in their theory of twisted level curves, which also works for more general group actions. On the other hand, inspired by M. Cornalba's theory of spin curves, E. Ballico, C. Casagrande and C. Fontanari gave a compactification of $\mathcal{R}_{g,2}$ in terms of *Prym curves* (see [BCF04]), which is related to Bernstein's compactification, but more accessible. After the study of moduli spaces of roots of line bundles by L. Caporaso, C. Casagrande and M. Cornalba in [CCC07], it became clear what the right definition of limits of level $\ell \geqslant 3$ curves should be. The study of moduli spaces of these *quasi-stable level ℓ curves* was initiated in [CEFS13]. This very convenient modular interpretation for the geometric points of $\overline{\mathcal{R}}_{g,\ell}$ is the one we are going to introduce here and use subsequently.

Definition 1.3. A *quasi-stable* curve of genus g is a connected nodal curve of arithmetic genus g such that every smooth rational component meets the rest of the curve in exactly two points, and these points belong to non-rational components. Such rational components are called *exceptional*.

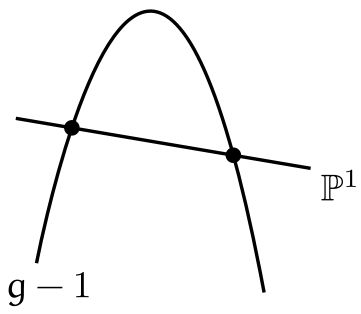

Figure 1.4: The prototypical quasi-stable curve.

Note that by blowing down all exceptional components of a quasi-stable curve we obtain a stable curve.

Definition 1.4. A *quasi-stable level ℓ curve* of genus g is a triple $[C, \eta, \beta]$ consisting of a quasi-stable curve C of genus g, a line bundle $\eta \in \mathrm{Pic}^0(C)$ and a sheaf homomorphism $\beta\colon \eta^{\otimes \ell} \to \mathcal{O}_C$, subject to the following conditions:

1. For each exceptional component E of C we have $\eta|_E = \mathcal{O}_E(1)$.
2. For each non-exceptional component the morphism β is an isomorphism.
3. For each exceptional component E and $\{p, q\} = E \cap \overline{C \setminus E}$ we have

$$\mathrm{ord}_p(\beta) + \mathrm{ord}_q(\beta) = \ell$$

A *family of quasi-stable level ℓ curves* over a scheme S is a triple $(\mathcal{C} \to S, \eta, \beta)$ where $\mathcal{C} \to S$ is a a flat family of quasi-stable curves, η is a line bundle on $\mathcal{C}$ and $\beta\colon \eta^{\otimes \ell} \to \mathcal{O}_{\mathcal{C}}$ is a sheaf homomorphism such that for each geometric fiber $C_s \to \{s\} \subset S$ the triple $(C_s, \eta|_{C_s}, \beta|_{C_s})$ is a quasi-stable level ℓ curve.

Quasi-stable level 2 curves are also called *Prym curves*. The fibered category of families of quasi-stable level ℓ curves defines a Deligne–Mumford stack whose associated coarse moduli space we denote by $\mathrm{Root}_{g,\ell}$. Since for $\ell > 3$ the singularities of $\mathrm{Root}_{g,\ell}$ are not normal, the definition of the actual moduli space $\overline{\mathcal{R}}_{g,\ell}$ is a bit more involved. It arises as a connected component of the coarse moduli space $\overline{\mathcal{M}}_g(B\,\mathbb{Z}_\ell)$ of twisted level curves ([ACV03]), which is a normalization of $\mathrm{Root}_{g,\ell}$. In particular the treatment of the universal curve over the Deligne–Mumford stack $\overline{\mathsf{R}}_{g,\ell}$ requires some further work. We direct the reader to the extensive discussions in [Chi08] and [CEFS13].

1.2.1 Boundary divisors

Let $\pi\colon \overline{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{M}}_g$ be the forgetful map. We study the boundary components of $\overline{\mathcal{R}}_{g,\ell}$. They lie over the boundary of $\overline{\mathcal{M}}_g$, so we can study the components lying over Δ_i for $i = 0, \dots, \lfloor \frac{g}{2} \rfloor$. Because of notational convenience sometimes boundary components of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{R}}_{g,\ell}$ will be denoted by the same symbols. However it should always be clear from the context which space we are considering.

The divisors $\boldsymbol{\Delta_i, \Delta_{g-i}, \Delta_{g:i}}$, $\mathbf{i \geq 1}$. First consider $i \geqslant 1$ and let $X \in \Delta_i$ be general, i.e., $X = C \cup D$ is the union of two curves of genera i and $g - i$ meeting transversally in a single node. The line bundle $\eta \in \mathrm{Pic}^0(X)$ on the corresponding level ℓ curve is determined by its restrictions $\eta_C = \eta|_C$ and $\eta_D = \eta|_D$ satisfying $\eta_C^{\otimes \ell} = \mathcal{O}_C$ and $\eta_D^{\otimes \ell} = \mathcal{O}_D$.

Either one of η_C and η_D (but not both) can be trivial, so $\pi^*(\Delta_i)$ splits into three irreducible components

$$\boldsymbol{\pi}^*(\Delta_i) = \Delta_i + \Delta_{g-i} + \Delta_{i:g-i}$$

where the general element in Δ_i is $[C \cup D, \eta_C \neq \mathcal{O}_C, \mathcal{O}_D]$, the generic point of Δ_{g-i} is of the form $[C \cup D, \mathcal{O}_C, \eta_D \neq \mathcal{O}_D]$ and the generic point of $\Delta_{i:g-i}$ looks like $[C \cup D, \eta_C \neq \mathcal{O}_C, \eta_D \neq \mathcal{O}_D]$. Observe that for $i = 1$ and $\ell \geqslant 3$, due to the extra automorphism on elliptic tails, we have the pullback formula $\pi^*(\Delta_1) = 2\Delta_1 + 2\Delta_{1:g-1} + \Delta_{g-1}$ and the map π is ramified along Δ_1 and $\Delta_{1:g-1}$.

The divisor Δ_0''. Now let $i = 0$. The generic point of Δ_0 in $\overline{\mathcal{M}}_g$ is a one-nodal irreducible curve C of geometric genus $g-1$. We first consider points of the form $[C, \eta]$ lying over C, i.e., without an exceptional component. Denote by $\nu\colon \widetilde{C} \to C$ the normalization and by p, q the preimages of the node. Then we have an exact sequence

$$0 \to \mathbb{C}^* \to \operatorname{Pic}^0(C) \xrightarrow{\nu^*} \operatorname{Pic}^0(\widetilde{C}) \to 0$$

which restricts to

$$0 \to \mathbb{Z}/\ell\mathbb{Z} \to \operatorname{Pic}^0(C)[\ell] \xrightarrow{\nu^*} \operatorname{Pic}^0(\widetilde{C})[\ell] \to 0$$

on the ℓ-torsion part. The group $\mathbb{Z}/\ell\mathbb{Z}$ represents the ℓ possible choices of gluing the fibers at p and q for each line bundle in $\operatorname{Pic}^0(\widetilde{C})[\ell]$. For the case $\nu^*\eta = \mathcal{O}_{\widetilde{C}}$ there are exactly $\ell - 1$ possible choices of $\eta \neq \mathcal{O}_C$. These curves $[C, \eta]$ correspond to the order ℓ analogues of the classical *Wirtinger double covers*

$$\widetilde{C}_1 \amalg \widetilde{C}_2/(p_1 \sim q_2, p_2 \sim q_1) \xrightarrow{2:1} \widetilde{C}/(p \sim q) = C$$

We denote by Δ_0'' the closure of the locus of level ℓ Wirtinger covers. Note that for $\ell > 3$ the divisor Δ_0'' is not irreducible. Indeed, up to switching the role of the points p and q lying over the node, the sections s of an ℓ-torsion line bundle $\eta' \in \operatorname{Pic}^0(\widetilde{C})$ that descend to C are determined by $s(p) = \xi^a s(q)$ where ξ is an ℓ-th root of unity and $1 \leqslant a \leqslant \ell - 1$. Hence we get precisely $\lfloor \ell/2 \rfloor$ irreducible components and each of them has order 2 over $\Delta_0 \subset \overline{\mathcal{M}}_g$.

The divisor Δ_0'. On the other hand, there are $\ell^{2(g-1)} - 1$ nontrivial elements in the group $\operatorname{Pic}^0(\widetilde{C})[\ell]$. For each of them there are ℓ choices of gluing, so we have a total of $\ell \cdot (\ell^{2g-2} - 1)$ choices for $\eta \in \operatorname{Pic}^0(C)$ such that $\nu^*\eta \neq \mathcal{O}_{\widetilde{C}}$. We let Δ_0' be the closure of the locus of pairs $[C, \eta]$ such that $\nu^*\eta \neq \mathcal{O}_{\widetilde{C}}$.

The divisors $\Delta_0^{(a)}$. We turn to the case of curves of the form $[X = \widetilde{C} \cup_{p,q} E, \eta]$ where E is an exceptional component. The stabilization of such a curve is again a one-nodal curve C. Denote by β the morphism $\eta^{\otimes \ell} \to \mathcal{O}_X$. Since $\eta|_E = \mathcal{O}_E(1)$, we must have $\beta_{E \setminus \{p,q\}} = 0$ and $\deg(\eta^{\otimes \ell}|_{\widetilde{C}}) = -\ell$. By swapping p and q if necessary, we can conclude that $\eta^{\otimes \ell}|_{\widetilde{C}} = \mathcal{O}_{\widetilde{C}}(-ap - (\ell - a)q)$ for some integer a with $1 \leqslant a \leqslant \lfloor \ell/2 \rfloor$. There are $\ell^{2(g-1)}$ choices of square roots of $\mathcal{O}_{\widetilde{C}}(-ap - (\ell - a)q)$ and each of these determines uniquely a Prym curve

$[X, \eta]$ of this form. We denote the closure of the locus of such curves by $\Delta_0^{(a)}$. Then the degree of $\Delta_0^{(a)}$ over Δ_0 is $2\ell^{2g-2}$ for all a. The factor 2 arises because of the symmetry in p and q.

1.3 Birational classification of projective varieties

Here we introduce the basic notions in the study of the birational geometry of moduli spaces. To any line bundle L on a projective variety X we can associate its graded ring of sections

$$R(X, L) = \bigoplus_{d=0}^{\infty} H^0(X, L^{\otimes d})$$

Definition 1.5. The *Iitaka dimension* $\kappa(X, L)$ of a line bundle L on X is defined to be $-\infty$ if $R(X, L) = 0$ and $\kappa(X, L) = \dim \operatorname{Proj} R(X, L)$ otherwise.

Definition 1.6. A line bundle L on X is called *big* if $\kappa(X, L) = \dim(X)$. Equivalently, the rational map from X to projective space induced by $L^{\otimes d}$ for $d \gg 0$ is birational onto its image.

There is a very useful equivalent condition for L to be big:

Lemma 1.7 (Kodaira's lemma; [Mat02, Lemma 6-2-7])**.** *A line bundle L is big if and only if it can be written as $L = A \otimes E$ where A is ample and E is effective. In particular, if L is big and D is an effective divisor then $L \otimes \mathcal{O}_X(D)$ is big as well.*

The case $L = K_X$, the canonical bundle of X, has its own terminology:

Definition 1.8. The *Kodaira dimension* $\kappa(X)$ of a smooth projective variety X is the Iitaka dimension of the canonical bundle, i.e., $\kappa(X) := \kappa(X, K_X)$.

The Kodaira dimension turns out to be a birational invariant of X, hence we can also define it for X only quasi-projective: just take $\kappa(X)$ to be $\kappa(\overline{X})$ for some compactification $\overline{X}$ of X. Furthermore, if X is singular, choose a desingularization $\widetilde{X} \to X$. The Kodaira dimension of X is then $\kappa(X) := \kappa(\widetilde{X})$ and is, as remarked, independent of the choice of $\widetilde{X}$. Note that we always have $-\infty \leqslant \kappa(X) \leqslant \dim(X)$. We say that X is of *general type* if K_X is big, i.e., if $\kappa(X) = \dim(X)$.

If $f \colon X \to Y$ is a finite cover, then the singularities of X may be very different from those of Y. Hence the question of determining $\kappa(X)$ can be hard, even if $\kappa(Y)$ is known. However, this poses no problem for the general type case:

Lemma 1.9 ([Kaw81, Corollary 9])**.** *Let $f \colon X \to Y$ be a generically surjective and generically finite morphism of algebraic varieties. Then $\kappa(X) \geqslant \kappa(Y)$.*

On the other end of the spectrum of Kodaira dimensions sit the following varieties:

Definition 1.10. A variety X of dimension n is called

- *rational* if there is a birational map $\mathbb{P}^n \dashrightarrow X$,
- *unirational* if there is a dominant rational map $\mathbb{P}^N \dashrightarrow X$ for some N,
- *uniruled* if for a general point in X there is a rational curve passing through it.

A rational variety is obviously unirational and a unirational variety is uniruled. The converses do not hold. It is also known that uniruledness implies $\kappa(X) = -\infty$. As a partial converse, the BDPP theorem ([BDPP13]) shows that X is uniruled if K_X is not pseudo-effective, i.e., it does not lie in the closure of the effective cone of X.

1.4 The Picard groups of moduli spaces of curves

A direct consequence of the construction of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{R}}_{g,\ell}$ is that the only singularities of these spaces are finite quotient singularities. It follows that all Weil divisors are actually $\mathbb{Q}$-Cartier. For this reason we will allow rational coefficients in all divisor classes we are considering.

On the other hand, the stacks $\overline{\mathsf{M}}_g$ and $\overline{\mathsf{R}}_{g,\ell}$ are smooth Deligne–Mumford stacks. The relation between the rational Picard group of $\overline{\mathsf{M}}_g$ and the coarse space $\overline{\mathcal{M}}_g$ is discussed in chapter 3.D of [HM98]. A rational divisor class on $\overline{\mathsf{M}}_g$ is defined as a map γ which associates to each family $\rho\colon \mathcal{X} \to B$ of stable curves a rational divisor class $\gamma(\rho) \in \mathrm{Pic}_{\mathbb{Q}}(B)$ on the base of the family. Additionally, γ is required to be functorial, i.e., given a fiber square

$$\begin{tikzcd}
\mathcal{X}' = B' \times_B \mathcal{X} \arrow[r] \arrow[d, "\rho'"'] & \mathcal{X} \arrow[d, "\rho"] \\
B' \arrow[r] & B
\end{tikzcd}$$

the class $\gamma(\rho')$ is required to be the pullback of $\gamma(\rho)$ under the map $B' \to B$. One can then show that γ is already determined by its values on all families over smooth, one-dimensional bases B.

It is very useful to observe that, at least with rational divisor classes, the Picard groups of the stack and the coarse moduli space are isomorphic:

Theorem 1.11 ([HM98, Proposition 3.88])**.**

$$\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g) \cong \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathsf{M}}_g)$$

In order to get a divisor class on the coarse space $\mathcal{M}_g$ from a divisor class on the stack, we can pass to a finite covering of $\overline{\mathcal{M}}_g$ where a universal family exists, construct the class there, and then push it forward to $\overline{\mathcal{M}}_g$ while dividing by the degree of the covering. Finite covers that can accomplish this have been put forward by E. Looijenga in [Loo92], for instance.

Remark 1.12. If we consider integral Picard groups, there exist nontrivial torsion classes in $\mathrm{Pic}_{\mathbb{Z}}(\overline{\mathcal{M}}_g)$.

The most important class in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$ is the *Hodge class* λ, the determinant of the *Hodge bundle* whose fiber over a smooth curve $[C]$ is the vector space of holomorphic differentials $H^0(C, K_C)$. To construct λ, consider a flat family $f\colon \mathcal{X} \to B$ of stable curves with relative dualizing sheaf ω_f. Then $\mathbb{E}_f = f_*\omega_f$ is the rank g Hodge bundle of this family and we set $\lambda(f) = \det(\mathbb{E}_f)$. It was J. Harer who showed that this class is already the whole picture on the space of smooth curves $\mathcal{M}_g$:

Theorem 1.13 ([Har83]). *We have* $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_g) = \mathbb{Q}[\lambda]$.

By δ_i we denote the class $[\Delta_i]_{\mathbb{Q}}$ of the boundary divisors Δ_i defined in section 1.1.1, where $i = 0, \dots, \lfloor g/2 \rfloor$. We also set $\delta = \sum_{i=0}^{\lfloor g/2 \rfloor} \delta_i$. The boundary classes, together with the Hodge class, generate the full Picard group of the moduli stack of stable curves:

Theorem 1.14 ([AC87]). *For any $g \geqslant 3$, the Picard group $\mathrm{Pic}(\overline{\mathsf{M}}_g)$ is freely generated over $\mathbb{Z}$ by λ and the boundary classes $\delta_0, \dots, \delta_{\lfloor g/2 \rfloor}$. The Picard group of M_g is freely generated by λ.*

A similar result is available for the moduli spaces of twisted level curves. Putting together our description of the boundary classes of $\overline{\mathcal{R}}_{g,\ell}$ and the results about the Picard groups of moduli spaces of curves with full level structure by A. Putman ([Put12]), we arrive at the following:

Theorem 1.15. *For $g \geqslant 5$ and $\ell \geqslant 2$ prime, the rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathsf{R}}_{g,\ell})$ is freely generated by λ and the classes δ_0', δ_0'', $\delta_0^{(a)}$, δ_i, δ_{g-i} and $\delta_{i:g-i}$ where $a = 1, \dots, \lfloor \ell/2 \rfloor$ and $i = 1, \dots, g-1$.*

Observe that we have two very natural divisor classes on $\overline{\mathcal{M}}_g$ which are in some sense opposites: the Hodge class λ and the class δ of the entire boundary. There is a very useful numerical invariant associated to an effective divisor D on $\overline{\mathcal{M}}_g$, namely its *slope* $s(D)$, which compares the λ and δ parts.

Definition 1.16. On the cone $\mathrm{Eff}(\overline{\mathcal{M}}_g)$ of pseudo-effective divisors on $\overline{\mathcal{M}}_g$ we define the *slope function* $s\colon \mathrm{Eff}(\overline{\mathcal{M}}_g) \to \mathbb{R} \cup \{\infty\}$ as follows:

$$s(D) = \inf \left\{ \frac{a}{b} \;\middle|\; a, b > 0 \text{ and } D \equiv a\lambda - b\delta - \sum_{i=0}^{\lfloor g/2 \rfloor} c_i \delta_i \text{ for some } c_i \geqslant 0 \right\}$$

If D is equivalent to a linear combination $a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$ with $a \geqslant 0$ and all $b_i \geqslant 0$ as well, then $s(D)$ is finite, otherwise we have $s(D) = \infty$. The *effective dichotomy lemma* says that the closure in $\overline{\mathcal{M}}_g$ of an effective divisor D on $\mathcal{M}_g$ always has slope $s(\overline{D}) < \infty$. In addition this means that

$$s(\overline{D}) = \frac{a}{\min_{i=0}^{\lfloor g/2 \rfloor} b_i}$$

1.5 Singularities of the moduli spaces of curves

In order to study the birational geometry of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{R}}_{g,\ell}$ we first have to better understand their singularities. As mentioned before, the construction of the moduli spaces directly shows that they have at worst finite quotient singularities, arising from curves with nontrivial automorphisms. Looking first at smooth curves, the locus of curves with extra automorphisms in $\mathcal{M}_g$ has fairly high codimension:

Lemma 1.17 ([Cor87])**.** *Assume* $g \geqslant 4$. *Every component* X *of the singular locus of* $\mathcal{M}_g$ *has*

$$\dim(X) \leqslant \dim(\mathcal{H}_g) = 2g - 1$$

with equality if and only if $X = \mathcal{H}_g$*, the locus of hyperelliptic curves.*

On the other hand, the moduli space $\overline{\mathcal{M}}_g$ of stable curves contains the boundary component Δ_1, where a general point is of the form $C \cup_p E$ with $[E, p]$ an elliptic curve. The involution -1 on the general elliptic tail $[E, p]$ is the only automorphism of the curve and does not in fact induce a singularity of $\overline{\mathcal{M}}_g$. However, if E is not general then its automorphism group is $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$, and in this case the point $[C \cup_p E]$ is indeed a singular point of $\overline{\mathcal{M}}_g$. Even worse, if $\mathrm{Aut}([E, p]) = \mathbb{Z}/6\mathbb{Z}$ then we get a non-canonical singularity of $\overline{\mathcal{M}}_g$ (see [HM82]). Summarizing, we have:

Lemma 1.18. *The singular locus of* $\overline{\mathcal{M}}_g$ *has codimension* 2*. The locus of non-canonical singularities of* $\overline{\mathcal{M}}_g$ *has codimension* 2 *as well.*

Although the singularities prevent the extension of locally defined canonical forms, they do not impose global adjunction conditions. In other words, if we have a canonical differential defined on the whole smooth part of $\overline{\mathcal{M}}_g$ then it extends over the singular locus. This was first proved by J. Harris and D. Mumford in their landmark paper on the Kodaira dimensions of $\overline{\mathcal{M}}_g$.

Theorem 1.19 ([HM82, §2])**.** *If* $g \geqslant 4$*, then for all* m*, every* m*-canonical form on* $\overline{\mathcal{M}}_g^{\mathrm{reg}}$ *extends to an* m*-canonical form on* $\overline{\mathcal{M}}_g$*. More precisely:*

$$H^0\left(\overline{\mathcal{M}}_g^{\mathrm{reg}}, K_{\overline{\mathcal{M}}_g^{\mathrm{reg}}}^{\otimes m}\right) \cong H^0\left(\widehat{\mathcal{M}}_g, K_{\widehat{\mathcal{M}}_g}^{\otimes m}\right)$$

for every desingularization $\widehat{\mathcal{M}}_g$ *of* $\overline{\mathcal{M}}_g$*.*

When trying to extend this result to $\overline{\mathcal{R}}_{g,\ell}$, no problem arises on the interior $\mathcal{R}_{g,\ell}$. In fact, $\mathcal{R}_{g,\ell}$ only has canonical singularities ([CF12]). But for $\ell \geqslant 5$, a new type of singularities arises from certain curves with at least three nonseparating nodes (Remark 2.43 loc. cit.). It is currently unclear whether a result similar to Theorem 1.19 holds in this case. Nevertheless, for $\ell = 2$ and $\ell = 3$ the singularities impose no global adjunction conditions:

Theorem 1.20 ([FL10, Theorem 6.1]; [CF12, Main Theorem])**.** *Fix $g \geqslant 4$ and $\ell = 2$ or $\ell = 3$. Let $\widehat{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{R}}_{g,\ell}$ be any desingularization. Then every pluricanonical form defined on the smooth locus $\overline{\mathcal{R}}_{g,\ell}^{\mathrm{reg}}$ of $\overline{\mathcal{R}}_{g,\ell}$ extends holomorphically to $\widehat{\mathcal{R}}_{g,\ell}$, that is, for all integers $m \geqslant 0$ we have isomorphisms*

$$H^0\left(\overline{\mathcal{R}}_{g,\ell}^{\mathrm{reg}}, K_{\overline{\mathcal{R}}_{g,\ell}^{\mathrm{reg}}}^{\otimes m}\right) \cong H^0\left(\widehat{\mathcal{R}}_{g,\ell}, K_{\widehat{\mathcal{R}}_{g,\ell}}^{\otimes m}\right)$$

1.6 The canonical classes of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{R}}_{g,\ell}$, and general type results

Since the singular locus of $\overline{\mathcal{M}}_g$ has codimension 2, we can define the canonical bundle $K_{\overline{\mathcal{M}}_g}$ as the unique extension of the canonical bundle on the smooth part of $\overline{\mathcal{M}}_g$. To calculate its class in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$ one considers first the canonical divisor class on the stack $\overline{\mathsf{M}}_g$. For a family $\rho\colon \mathcal{X} \to B$ of stable curves we let Ω_ρ be the sheaf of relative Kähler differentials and ω_ρ be the relative dualizing sheaf. Then we set

$$K_{\overline{\mathsf{M}}_g}(\rho) = c_1\big(\rho_*(\Omega_\rho \otimes \omega_\rho)\big)$$

Using the Grothendieck–Riemann–Roch formula we can calculate the expansion of $K_{\overline{\mathsf{M}}_g}$ in terms of the standard basis of $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathsf{M}}_g)$:

Theorem 1.21 ([HM82])**.** *The canonical class of $\overline{\mathsf{M}}_g$ is*

$$K_{\overline{\mathsf{M}}_g} = 13\lambda - 2\delta$$

A Riemann–Hurwitz type formula for the ramification of the coarsening map $f\colon \overline{\mathsf{M}}_g \to \overline{\mathcal{M}}_g$ allows us to deduce a formula for the canonical class of $\overline{\mathcal{M}}_g$ when $g \geqslant 4$. The only difference is an additional summand of $-\delta_1$, coming from the simple ramification of f along Δ_1 due to the involution on elliptic tails of general curves in Δ_1.

Corollary 1.22. *For $g \geqslant 4$ the canonical class $K_{\overline{\mathcal{M}}_g}$ of the coarse moduli space $\overline{\mathcal{M}}_g$ is*

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\sum_{i=2}^{\lfloor g/2 \rfloor} \delta_i$$

These results lead us to the following strategy to prove that $\mathcal{M}_g$ is of general type. If we can find an effective divisor E in $\overline{\mathcal{M}}_g$ with

$$E \equiv a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i$$

such that

$$\frac{a}{b_i} < \frac{13}{2} \quad \text{for } i \neq 1, \quad \frac{a}{b_1} < \frac{13}{3} \tag{1.1}$$

then it follows that we can write

$$K_{\overline{\mathcal{M}}_g} \equiv \varepsilon\lambda + \alpha E + \beta D$$

where D is supported on the boundary, $\alpha, \beta \geqslant 0$ and $\varepsilon > 0$. Since the Hodge class λ is big, $K_{\overline{\mathcal{M}}_g}$ must be big as well. If we replace the strict inequalities in (1.1) by their non-strict counterparts, we would still be able to prove that $K_{\overline{\mathcal{M}}_g}$ was effective, hence $\kappa(\mathcal{M}_g) \geqslant 0$. This method was first envisioned in [HM82] and has been the strategy of choice ever since. Using the definition of the slope $s(D)$ of an effective divisor D we get the following much more memorizable result.

Lemma 1.23. *If there exists an effective divisor D on $\overline{\mathcal{M}}_g$ with $s(D) < \frac{13}{2}$ then $\overline{\mathcal{M}}_g$ is of general type.*

Interestingly, at least for $g \leqslant 23$ the only relevant data for determining whether $\overline{\mathcal{M}}_g$ is of general type are the coefficients of λ and δ_0 in the classes of effective divisors:

Theorem 1.24 ([FP05, Theorem 1.4]). *Let $s_g = \min\{s(D) \mid D \in \mathrm{Eff}(\overline{\mathcal{M}}_g)\}$ be the slope of $\overline{\mathcal{M}}_g$. For any $g \leqslant 23$ there exists $\varepsilon_g > 0$ such that for any effective divisor D on $\overline{\mathcal{M}}_g$ with $s_g \leqslant s(D) \leqslant s_g + \varepsilon_g$ we have $s(D) = a/b_0$, i.e., $b_0 \leqslant b_i$ for all $i \geqslant 1$.*

Conjecturally this result is true in every genus.

The previous considerations all hold in a similar form for $\mathcal{R}_{g,\ell}$. First, using the Hurwitz formula for the maps $\pi\colon \overline{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{M}}_g$ one can immediately deduce:

Theorem 1.25 ([FL10; CEFS13]). *For $g \geqslant 4$, the canonical class of $\overline{\mathcal{R}}_{g,2}$ is*

$$K_{\overline{\mathcal{R}}_g} = 13\lambda - 2(\delta_0' + \delta_0'') - 3\delta_0^{(1)} - 2\sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} (\delta_i + \delta_{g-i} + \delta_{i:g-i}) - (\delta_1 + \delta_{g-1} + \delta_{1:g-1})$$

In the case of higher level curves we have the following expression for $K_{\overline{\mathcal{R}}_{g,\ell}}$:

$$K_{\overline{\mathcal{R}}_{g,\ell}} = 13\lambda - 2(\delta_0' + \delta_0'') - (\ell+1)\sum_{k=1}^{\lfloor \ell/2 \rfloor} \delta_0^{(k)} - 2\sum_{i=1}^{\lfloor g/2 \rfloor} (\delta_i + \delta_{g-i} + \delta_{i:g-i}) - \delta_{g-1}$$

As can be seen from the coefficients of the divisor classes in these expressions, on $\overline{\mathcal{R}}_{g,\ell}$ we do not have a concept of slope of an effective divisor that is similarly useful as that on $\overline{\mathcal{M}}_g$. More to the point, we would need two different slopes: one for the ratio of λ to $\delta_0' + \delta_0''$, and one for the ratio of λ to $\sum \delta_0^{(a)}$. In practice, it turns out that one seldomly finds naturally defined effective divisors minimizing both ratios at the same time. When trying to prove that $\overline{\mathcal{R}}_{g,\ell}$ is of general type for some g and ℓ, one usually looks for two different effective divisors: one having $a/b_0' < 13/2$ and one having $a/b_0^{(1)} < 13/(\ell+1)$. Some effective linear combination of these two then hopefully yields the desired result.

Implicit in the previous discussion is a result similar to Theorem 1.24, saying that we can restrict our attention to the boundary divisors δ_0', δ_0'' and $\delta_0^{(k)}$ of irreducible nodal curves:

Lemma 1.26 ([CEFS13, Remark 3.5])**.** *Let $g \leqslant 23$ and $\ell \geqslant 2$. In order to prove that $K_{\overline{\mathcal{R}}_{g,\ell}}$ is effective (respectively big) it is enough to exhibit an effective divisor*

$$E \equiv a\lambda - b_0'\delta_0' - b_0''\delta_0'' - \sum_{k=1}^{\lfloor \ell/2 \rfloor} b_0^{(k)}\delta_0^{(k)} - \sum_{i=1}^{\lfloor g/2 \rfloor} (b_i\delta_i + b_{g-i}\delta_{g-i} + b_{i:g-i}\delta_{i:g-i})$$

with $a/b_0' \leqslant 13/2$, $a/b_0'' \leqslant 13/2$ and $a/b_0^{(k)} \leqslant 13/(\ell+1)$ for all $k = 1, \ldots, \lfloor \ell/2 \rfloor$ (respectively $<$ instead of $\leqslant$). The coefficients b_i, b_{g-i} and $b_{i:g-i}$ are then automatically suitably bounded.

1.7 Brill–Noether theory

1.7.1 Brill–Noether theory on a fixed curve

A natural source of effective divisors on $\overline{\mathcal{M}}_g$ are loci of curves possessing line bundles with an unusual amount of global sections compared to their degree. One typical example is the locus of d-gonal curves, i.e., curves which are $d : 1$ ramified covers of the projective line. These loci are usually of high codimension, but there are numerical ways to isolate the cases where they form divisors.

By a $\mathfrak{g}^r_d$ we mean a linear series of degree d and dimension r, i.e., the projectivization of an $(r+1)$-dimensional vector subspace of the global sections $H^0(C, L)$ of a line bundle L of degree d. For a curve $[C] \in \mathcal{M}_g$ we let

$$W^r_d(C) = \left\{ L \in \mathrm{Pic}^d(C) \;\middle|\; h^0(C, L) \geqslant r+1 \right\}$$

and

$$G^r_d(C) = \left\{ \mathfrak{g}^r_d\text{'s on } C \right\}$$

which, as a set, is equal to

$$\left\{ (L, V) \;\middle|\; L \in \mathrm{Pic}^d(C), V \subseteq H^0(C, L), \dim V = r+1 \right\}$$

Furthermore, let $\mathcal{M}^r_{g,d}$ denote the locus of curves $[C] \in \mathcal{M}_g$ where $W^r_d(C) \neq \emptyset$.

The subject and central question of Brill–Noether theory then is the following: Given positive integers g, r and d, for which curves of genus g is $W^r_d(C)$ nonempty? In other words, how can we describe the locus $\mathcal{M}^r_{g,d}$? Since a base point free $\mathfrak{g}^r_d$ is the same as a degree d map $C \to \mathbb{P}^r$, Brill–Noether theory is closely related to studying the ways in which a curve can be embedded in projective space.

Many of the basic questions, in particular for general curves, are answered by a fairly simple invariant associated to the triple (g, r, d):

Definition 1.27. The *Brill–Noether number* $\rho(g, r, d)$ is

$$\rho(g, r, d) = g - (r + 1)(g - d + r)$$

The Brill–Noether number completely controls the existence of $\mathfrak{g}^r_d$'s on the general curve. This is the content of the Brill–Noether theorem, formulated already in the second half of the 19th century by A. von Brill and M. Noether ([BN74]) and proven rigorously by P. Griffiths and J. Harris in 1980:

Theorem 1.28 ([GH80])**.** *A general curve of genus g has a $\mathfrak{g}^r_d$ if and only if the inequality $\rho(g, r, d) \geqslant 0$ holds.*

This means that $\mathcal{M}^r_{g,d} = \mathcal{M}_g$ if and only if $\rho(g, r, d) \geqslant 0$. If $\rho(g, r, d) < 0$ then $\mathcal{M}^r_{g,d}$ has codimension at least 1. In this case the next question to ask is about the precise codimension of $\mathcal{M}^r_{g,d}$. F. Steffen proved in [Ste98] that locally the locus $\mathcal{M}^r_{g,d}$ can be given a determinantal description, i.e., it can be written as the degeneracy locus between vector bundles of the same rank. Therefore its codimension cannot exceed $-\rho(g, r, d)$ unless it is empty.

Theorem 1.29 ([Ste98, Theorem 0.1])**.** *If $\mathcal{M}^r_{g,d} \neq \emptyset$ then every irreducible component of $\mathcal{M}^r_{g,d}$ has codimension at most* $\max\{0, -\rho(g, r, d)\}$.

In the opposite direction the general picture is not yet completely clear. One important bound can be found in [EH89]:

Theorem 1.30 ([EH89, Theorem 1.1])**.** *If $\rho(g, r, d) \leqslant -2$, any component of $\mathcal{M}^r_{g,d}$ has codimension at least two.*

In the most interesting cases $\rho(g, r, d)$ will be close to zero and we have more information available. Combining Steffen's theorem with results by Eisenbud–Harris ([EH89]) and D. Edidin ([Edi93]), we get the following result for $\rho \geqslant -3$:

Theorem 1.31 ([Ste98, Theorem 0.2])**.**

i) If $\rho = -1$, then $\mathcal{M}^r_{g,d}$ is an irreducible divisor.

ii) If $\rho = -2$, then every irreducible component of $\mathcal{M}^r_{g,d}$ has codimension 2.

iii) If $\rho = -3$ and $g \geqslant 12$, then every irreducible component of $\mathcal{M}^r_{g,d}$ has codimension 3.

Given a triple (g, r, d) and a curve C such that $W^r_d(C) \neq \emptyset$, one can give $W^r_d(C)$ and $G^r_d(C)$ the structure of a determinantal variety. We can then ask about their dimension, connectedness, irreducibility and smoothness. For the general curve these questions are all answered in full by a series of theorems that we now quickly discuss. We start with the connectedness result of W. Fulton and R. Lazarsfeld:

Theorem 1.32 ([FL81]). *If $\rho(g, r, d) \geqslant 1$, then for every curve $[C] \in \mathcal{M}_g$ the varieties $G^r_d(C)$ and $W^r_d(C)$ are connected.*

Connectedness cannot be expected to hold in the case $\rho(g, r, d) = 0$ since $W^r_d(C)$ on the general curve will consist of a finite number of points. The precise number was first computed by G. Castelnuovo, and a derivation can be found in [ACGH85, Chapter VII].

Theorem 1.33 ([ACGH85, Theorem V.1.3]). *If $\rho(g, r, d) = 0$, then on the general curve $W^r_d(C)$ consists of precisely*

$$g! \cdot \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}$$

points.

More generally, the dimension of $G^r_d(C)$ on the general curve is given by $\rho(g, r, d)$:

Theorem 1.34 ([GH80]). *Let $\rho(g, r, d) \geqslant 0$ and $[C] \in \mathcal{M}_g$ be general. Then $G^r_d(C)$ is reduced and of pure dimension $\rho(g, r, d)$.*

For arbitrary curves we still have $\rho(g, r, d)$ as a lower bound for the dimension of every irreducible component of $G^r_d(C)$.

Going back to the general curve, the smoothness problem was solved by D. Gieseker:

Theorem 1.35 ([Gie82]). *If $[C] \in \mathcal{M}_g$ is general, then $G^r_d(C)$ is smooth of dimension $\rho(g, r, d)$.*

As an immediate corollary of the Connectedness and the Smoothness Theorem we obtain that for $[C] \in \mathcal{M}_g$ general and $\rho(g, r, d) \geqslant 1$, both loci $G^r_d(C)$ and $W^r_d(C)$ are irreducible.

Smoothness of $W^r_d(C)$ at a point L is controlled by the *Petri map*

$$\mu \colon H^0(C, L) \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)$$

whose kernel dimension computes the difference of the dimensions of $W^r_d(C)$ and its tangent space at L. Hence a reformulation of Theorem 1.35 is

Theorem 1.36. *On a general curve the Petri map is injective for every line bundle* L.

This formulation is very useful and is often called the Gieseker–Petri Theorem. We now define the prototypical divisors that have been used in the study of the Kodaira dimensions of $\mathcal{M}_g$:

Definition 1.37 (Brill–Noether and Gieseker–Petri divisors)**.** We let

$$\mathcal{GP}_g = \left\{[C] \in \mathcal{M}_g \;\middle|\; C \text{ does not satisfy the Gieseker–Petri Theorem}\right\}$$

The divisorial components of $\mathcal{GP}_g$ are called *Gieseker–Petri divisors*. Furthermore, in the case $\rho(g, r, d) = -1$, the locus $\mathcal{M}^r_{g,d}$ is an irreducible divisor by Theorem 1.31, called a *Brill–Noether divisor*.

1.7.2 Brill–Noether theory on a moving curve

Combining all the varieties $W^r_d(C)$ and $G^r_d(C)$ when [C] ranges over all smooth curves in $\mathcal{M}_g$, we obtain what are called the universal Brill–Noether varieties $\mathcal{W}^r_d$ and $\mathfrak{G}^r_d$ over $\mathcal{M}_g$. They can indeed be constructed as algebraic varieties, which is described in detail in [ACG11, Chapter XXI]. As is the case with $\mathcal{M}_g$, they also have stacky incarnations which are useful to work with if universal families are needed.

Here we want to summarize what is known in general about their dimension, connectedness, irreducibility and smoothness. First we present a lower bound on the dimension of every irreducible component:

Theorem 1.38 ([ACG11, Proposition XXI.3.21])**.** *Every irreducible component of* $\mathfrak{G}^r_d$ *has dimension at least* $3g - 3 + \rho(g, r, d)$. *If* $r \geqslant g - d$, *the same holds for every irreducible component of* $\mathcal{W}^r_d$.

Note that the variety $\mathfrak{G}^r_d$ may well be empty, even if $3g - 3 + \rho(g, r, d)$ is nonnegative.

The next fairly straightforward result says that smoothness of $G^r_d(C)$ for the general curve translates to smoothness at the corresponding points of $\mathfrak{G}^r_d$:

Proposition 1.39 ([ACG11, Corollary XXI.5.31])**.** *If a curve* $[C] \in \mathcal{M}_g$ *satisfies the Gieseker–Petri theorem (e.g. if* [C] *is general) then* $\mathfrak{G}^r_d$ *is smooth of dimension* $3g - 3 + \rho(g, r, d)$ *along* $G^r_d(C)$. *Furthermore, the points of* $W^r_d(C)$ *along which* $\mathcal{W}^r_d$ *is singular are precisely those belonging to* $\mathcal{W}^{r+1}_d$. *In particular,* $\mathcal{W}^r_d$ *is smooth of dimension* $3g - 3 + \rho(g, r, d)$ *along* $W^r_d(C) \setminus W^{r+1}_d(C)$.

For $r = 1$ more can be said:

Proposition 1.40 ([ACG11, Proposition XXI.6.8])**.** *If* $2 \leqslant d \leqslant g + 1$ *then* $\mathfrak{G}^1_d$ *is smooth, the singular locus of* $\mathcal{W}^1_d$ *is* $\mathcal{W}^2_d$ *and*

$$\dim \mathfrak{G}^1_d = \dim \mathcal{W}^1_d = 3g - 3 + \rho(g, r, d)$$

The spaces of one-dimensional linear series (i.e. maps to $\mathbb{P}^1$) are called *Hurwitz spaces* and they turn out to be irreducible. Consider first the space $\mathcal{H}_{d,g}$ of simply branched covers $C \to \mathbb{P}^1$ where C is a genus g curve. By associating to a cover its set of branch points we get a map

$$\mathcal{H}_{d,g} \to ((\mathbb{P}^1)^d \setminus \Delta)/S_d$$

where S_d is the symmetric group. This map is étale by the Riemann existence theorem. Hence $\mathcal{H}_{d,g}$ is smooth. Its irreducibility follows from the famous connectedness theorem of A. Clebsch and J. Lüroth:

Theorem 1.41 ([Cle73]). *$\mathcal{H}_{d,g}$ is connected.*

Since one can construct $\mathfrak{G}^1_d$ as a partial compactification of $\mathcal{H}_{d,g}$, it is irreducible as well.

Things become more complicated as soon as $r > 1$. However, if we assume $\rho(g, r, d) \geqslant 1$, then it follows from the Gieseker–Petri Theorem 1.35, and the fact that a general such $\mathfrak{g}^r_d$ corresponds to an embedding ([EH83]), that there is a unique irreducible component of $\mathfrak{G}^r_d$ dominating $\mathcal{M}_g$.

D. Eisenbud and J. Harris succeeded in proving a similar statement in the cases $\rho(g, r, d) = 0$ and $\rho(g, r, d) = -1$:

Theorem 1.42 ([EH87a]). *If $\rho(g, r, d) = 0$, then there is a unique irreducible component of $\mathfrak{G}^r_d$ dominating $\mathcal{M}_g$.*

Theorem 1.43 ([EH89]). *If $\rho(g, r, d) = -1$, there is a unique irreducible component of the variety $\mathfrak{G}^r_d$ whose image in $\mathcal{M}_g$ is of codimension one.*

Finally, in the case $r = 2$ and $\rho(g, 2, d) = 0$ we observe that for the general triple $[C, V]$ in a component of $\mathfrak{G}^2_d$ the curve C is birationally mapped by V to a nodal plane curve. Since the Severi variety of irreducible plane curves of degree d and arithmetic genus g is irreducible ([Har86]), $\mathfrak{G}^2_d$ is irreducible as well.

Other general results are currently not known. Proving irreducibility for a particular triple (g, r, d) can also be quite challenging and often entails detailed analysis of the loci $\mathcal{M}^r_{g,d}$ and the Brill–Noether theory of all curves $[C] \in \mathcal{M}^r_{g,d}$, not just the general ones.

1.8 The Kodaira dimensions of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{R}}_{g,\ell}$

The Kodaira dimension of $\mathcal{M}_g$ has a huge impact on possible parametrizations of curves of genus g. If $\mathcal{M}_g$ is unirational then we can essentially get a general curve of genus g by varying free parameters in some equations, subject to some inequalities. On the other hand, if $\kappa(\mathcal{M}_g) \geqslant 0$ (so in particular if $\mathcal{M}_g$ is of general type) then a general curve of genus g is not the hyperplane section of a non-ruled surface S:

Lemma 1.44. *Suppose $\mathcal{M}_g$ is not uniruled and let S be any surface containing C, such that C moves in its linear system $|\mathcal{O}_S(C)|$. Then S is birational to $C \times \mathbb{P}^1$.*

The history of the problem of determining the Kodaira dimension of $\mathcal{M}_g$ is in itself very interesting. We start by summing up what is currently known about $\kappa(\mathcal{M}_g)$:

Theorem 1.45.

- $\mathcal{M}_g$ *is rational for* $2 \leqslant g \leqslant 6$,
- $\mathcal{M}_g$ *is unirational for* $7 \leqslant g \leqslant 14$,
- $\mathcal{M}_{15}$ *is rationally connected,*
- $\mathcal{M}_{16}$ *is uniruled,*
- $\kappa(\mathcal{M}_{23}) \geqslant 2$,
- $\mathcal{M}_g$ *is of general type for* $g = 22$ *and* $g \geqslant 24$.

The first result in this direction was given by F. Severi more than a century ago ([Sev15]). He proved that $\mathcal{M}_g$ is unirational for $g \leqslant 10$ by considering, for the general curve, nodal models of minimal degree d in $\mathbb{P}^2$. The nodes can be chosen to be in general position precisely up to genus 10. Led on by this success, Severi also conjectured that perhaps $\mathcal{M}_g$ should be unirational for all g. In the following decades this conjecture generally seems to have been considered plausible.

Severi's method fails for $g \geqslant 11$ and it was not until 1981 that E. Sernesi was able to prove the unirationality of $\mathcal{M}_{12}$ in [Ser81]. In the meantime, it was shown by J.-I. Igusa in [Igu60] that $\mathcal{M}_2$ is in fact a rational variety, strenghtening Severi's result. Subsequently, unirationality of $\mathcal{M}_{11}$ and $\mathcal{M}_{13}$ was proved by M.-C. Chang and Z. Ran ([CR84]). The same authors later proved that $\mathcal{M}_{15}$ ([CR86]) and $\mathcal{M}_{16}$ ([CR91]) have Kodaira dimension $-\infty$. These proofs were notable because they were intersection-theoretic in nature and did not give any effective construction of a ruled parameter space. In 2005, A. Verra showed unirationality for $\mathcal{M}_{14}$ in [Ver05]. Furthermore, he and A. Bruno gave a proof that $\mathcal{M}_{15}$ is rationally connected ([BV05]). It was later noted by G. Farkas ([Far10, Theorem 2.7]) that in light of the BDPP theorem ([BDPP13]) the results of Chang and Ran in [CR91] actually imply that $\mathcal{M}_{16}$ is uniruled. As for rationality, N. I. Shepherd-Barron proved in 1987 that $\mathcal{M}_4$ is rational ([She87]) and in 1989 that the same is true for $\mathcal{M}_6$ ([She89]). This was followed by results of P. I. Katsylo, showing that $\mathcal{M}_5$ ([Kat92]) and $\mathcal{M}_3$ ([Kat96]) are rational as well.

Contrary to what was expected in light of Severi's conjecture, J. Harris and D. Mumford proved in 1982 that $\mathcal{M}_g$ is of general type for all odd $g \geqslant 25$ ([HM82]). In the same paper they obtained the bound $\kappa(\mathcal{M}_{23}) \geqslant 0$. This was followed by [Har84], showing that also for even $g \geqslant 40$ we have $\mathcal{M}_g$ of general type. With their theory of limit linear series (developed in [EH86]), D. Eisenbud and J. Harris were then able to prove that $\mathcal{M}_g$ is of general type for all $g \geqslant 24$ and they also showed $\kappa(\mathcal{M}_{23}) \geqslant 1$ ([EH87b]). Finally, G. Farkas proved the bound $\kappa(\mathcal{M}_{23}) \geqslant 2$ in [Far00] and settled the case for $\mathcal{M}_{22}$ in [Far10] by proving it is of general type as well.

We now come to the case of level curves. So far, Prym varieties (i.e. level $\ell = 2$) have gotten the most attention from the mathematical community. First we again summarize what is the state of the art here:

Theorem 1.46. *For $\ell = 2$, the moduli space $\mathcal{R}_{g,2}$ of Prym varieties is*

- *rational for $g \leqslant 4$,*
- *unirational for $5 \leqslant g \leqslant 7$,*
- *uniruled for $g = 8$,*
- *of nonnegative Kodaira dimension for $g = 12$, i.e., $\kappa(\mathcal{R}_{12,2}) \geqslant 0$,*
- *of general type for $g \geqslant 14$.*

The fact that $\mathcal{R}_{2,2}$ is rational is classical and a short proof can be found in [Dol08]. Next came the proof by F. Catanese that $\mathcal{R}_{4,2}$ is rational (see [Cat83]). Three different proofs of the fact that $\mathcal{R}_{6,2}$ is unirational were published almost at the same time by R. Donagi ([Don84]), by S. Mori and S. Mukai ([MM83]), and by A. Verra ([Ver84]). Although F. Catanese and I. V. Dolgachev announced proofs of the rationality of $\mathcal{R}_{3,2}$ in the 1980s, the earliest published result is by P. I. Katsylo in [Kat94]. A more recent result is the proof that $\mathcal{R}_{5,2}$ is unirational, given at around the same time by the group of E. Izadi, M. Lo Giudice and G. K. Sankaran ([ILS09]), and by A. Verra ([Ver08]). After G. Farkas and A. Verra showed in [FV12] that $\mathcal{R}_{7,2}$ is uniruled, they were able to improve this result to unirationality in [FV16] and furthermore show that $\mathcal{R}_{8,2}$ is uniruled.

The study of the birational geometry of $\mathcal{R}_{g,2}$ in the range where these spaces are of general type was initiated by G. Farkas and K. Ludwig in [FL10]. They were able to prove that $\mathcal{R}_{g,2}$ is of general type for $g = 14$ and $g \geqslant 16$, and with the same method they also gave the lower bound $\kappa(\mathcal{R}_{12,2}) \geqslant 0$. The genus 15 case will be treated in this thesis with the result that $\mathcal{R}_{15,2}$ is also shown to be of general type (see chapter 2).

Finally, we come to the case of level 3 curves. The results that we have available today are listed in the following theorem:

Theorem 1.47. *In the case $\ell = 3$ the moduli space $\mathcal{R}_{g,3}$ is*

- *rational for $g = 3$,*
- *unirational for $g = 4$,*

- *of Kodaira dimension* $\kappa(\mathcal{R}_{11,3}) \geqslant 19$ *for* $g = 11$,
- *of general type for* $g = 8$ *and* $g \geqslant 12$.

The rationality of $\mathcal{R}_{3,3}$ was established by I. Bauer and F. Catanese in [BC10]. For genus 4, it is known that the moduli space $\mathcal{R}_{4,\langle 3\rangle}$ of curves together with an order 3 subgroup of their Jacobian is rational (see [BV10]). $\mathcal{R}_{4,3}$ is a finite cover of $\mathcal{R}_{4,\langle 3\rangle}$ and by [BV10, Theorem 4.2] we have $\mathcal{R}_{4,3} \cong \mathcal{P} \times \mathbb{P}^5$, where $\mathcal{P}$ is the moduli space of six points in $\mathbb{P}^2$. It is known that $\mathcal{P}$ is unirational, hence $\mathcal{R}_{4,3}$ is unirational as well.

In the other direction, A. Chiodo, D. Eisenbud, G. Farkas and F.-O. Schreyer proved in [CEFS13] that $\mathcal{R}_{g,3}$ is of general type as soon as $g \geqslant 12$. They also obtained the bound $\kappa(\mathcal{R}_{11,3}) \geqslant 19$. One of the aims of this thesis is to show that $\mathcal{R}_{8,3}$ is of general type as well (see chapter 3).

Instead of just asking for the Kodaira dimension of $\mathcal{R}_{g,\ell}$, one could also focus on the spaces $\mathcal{R}_{g,\langle\ell\rangle}$ just introduced, as well as on $\mathcal{R}_{g,[\ell]}$, the moduli space of curves with full level structure, i.e., curves together with a basis of the ℓ-torsion of their Jacobian. In the first case, heuristics suggest that the Kodaira dimensions of $\mathcal{R}_{g,\ell}$ and $\mathcal{R}_{g,\langle\ell\rangle}$ do not exhibit very different behavior. The transition point from unirationality to general type is however still a mystery in both cases and one can expect interesting results.

On the other hand, for the moduli spaces $\mathcal{R}_{g,[\ell]}$ of curves with full level structure, Mumford proves in [Mum77] that they are of log general type for $\ell \geqslant 3$. Although this does not settle the question for which g and ℓ they are actually of general type, results about moduli spaces of abelian varieties of low dimension (where we have the Torelli map to compare them to moduli spaces of curves) suggest that there are not many interesting phenomena to be discovered. To be precise, while J. A. Todd proves in [Tod36] that $\mathcal{A}_2[3]$ is rational, T. Yamazaki ([Yam76]) and W. Wang ([Wan93]) show that $\mathcal{A}_2[\ell]$ is of general type for all $\ell \geqslant 4$. For abelian threefolds (or curves of genus 3) we have that $\mathcal{A}_3[\ell]$ is of general type for $\ell \geqslant 3$ by an argument of K. Hulek ([Hul00]). These results might explain why up to now there has been no published interest in studying the Kodaira dimensions of $\mathcal{R}_{g,[\ell]}$ systematically. For a more in-depth discussion of the birational geometry of moduli spaces of abelian varieties with level structure and an extensive list of references, see the survey [HS02].

1.9 Kernel bundles

Up to this point we have been talking about moduli spaces of curves with or without level structure. This is sufficient background material for Chapters 2 and 3. However, in order to study the normal bundles of canonical curves in Chapter 4, we need some more preliminary results on vector bundles on curves.

Perhaps the most important tool are kernel bundles, useful in estimating dimensions of global sections of vector bundles, as well as in studying multiplication maps of these sections. As an example, D. C. Butler used them in [But94] to study the normal generation of certain vector bundles on curves. Kernel bundles have also been used extensively in the context of Koszul cohomology, for instance in construction of Koszul divisors on moduli spaces of curves (see [Far09]). R. Lazarsfeld used them to show that a general curve on a general K3 surface satisfies the Brill–Noether–Petri theorem [Laz86]. C. Voisin made use of Kernel bundles in [Voi02; Voi05] to prove Green's conjecture for generic curves of even and odd genus, respectively.

We start with a definition.

Definition 1.48. Let L be a line bundle on a curve C. The *kernel bundle* M_L of L is defined to be the kernel of the evaluation map

$$H^0(C, L) \otimes \mathcal{O}_C \to L \tag{1.2}$$

Hence M_L sits in the exact sequence

$$0 \to M_L \to H^0(C, L) \otimes \mathcal{O}_C \to L$$

As a subbundle of the (free) vector bundle $H^0(C, L) \otimes \mathcal{O}_C$, the kernel bundle M_L is itself a vector bundle. We denote its dual by $Q_L = M_L^\vee$. If L is globally generated, then the evaluation map in (1.2) is surjective and the sequence

$$0 \to M_L \to H^0(C, L) \otimes \mathcal{O}_C \to L \to 0$$

is also exact on the right. We obtain the dual short exact sequence

$$0 \to L^{-1} \to H^0(C, L)^\vee \otimes \mathcal{O}_C \to Q_L \to 0$$

which shows that Q_L is globally generated. Taking global sections in this exact sequence also shows $h^0(C, Q_L) \geqslant h^0(C, L)$ and we expect equality to hold. We furthermore see that $\mathrm{rk}(Q_L) = h^0(C, L) - 1$ and $\det Q_L = L$.

The following two results will be used later on. Their proofs are provided here, since we could not find them in the literature.

Lemma 1.49. *Let L be a base point free line bundle and Q_L the dual of its kernel bundle. Let $A \hookrightarrow L$ be a base point free sub-line bundle. Then there is an exact sequence*

$$0 \to F \to Q_L \to Q_A \to 0$$

with a vector bundle F.

Proof. We write down the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_A \arrow[r] \arrow[d, hook] & \mathcal{O}_C \otimes H^0(C,A) \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & M_L \arrow[r] & \mathcal{O}_C \otimes H^0(C,L) \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

with exact rows and vertical inclusions. Using the Snake Lemma we get an exact sequence of cokernels

$$0 \to E \to \mathcal{O}_C \otimes (H^0(C,L)/H^0(C,A)) \to L/A \to 0$$

Since $\mathcal{O}_C \otimes (H^0(C,L)/H^0(C,A))$ is a vector bundle, E is as well. Thus we get an exact sequence

$$0 \to M_A \to M_L \to E \to 0$$

which upon dualizing and letting $F = E^\vee$ leads to the claim. ∎

Lemma 1.50. *Let A be a line bundle on C with $h^0(C,A) = 2$ and fix an integer r. Assume $h^0(C,A^{\otimes j}) - h^0(C,A^{\otimes(j-1)}) = 1$ for all $j \leqslant r$. Then the dual kernel bundle of $L = A^{\otimes r}$ splits as $Q_L = A^{\oplus r}$.*

Proof. Use the previous Lemma 1.49 inductively and observe that the short exact sequence splits at every step. ∎

The bundles Q_L are often stable and in many cases we have a natural isomorphism $H^0(C,L)^\vee \cong H^0(C,Q_L)$. In this way, kernel bundles provide many examples of stable bundles with unusually many global sections. They have therefore been used to construct vector bundles with special properties on curves and surfaces, e.g., Ulrich bundles ([ES03]), and bundles violating Mercat's conjecture ([FO12]). In this note they will play a central role in the proof of the stability of the normal bundle of canonical genus 8 curves.

Note that we can also define the kernel bundle M_E and its dual Q_E for any vector bundle E on C. If E is globally generated, we have the numerical facts $\mathrm{rk}(M_E) = h^0(C,E) - \mathrm{rk}(E)$, $h^0(C,Q_E) \geqslant h^0(C,E)$ and $\det(Q_E) = \det(E)$. Furthermore, Q_E is globally generated as well.

1.10 Curves and morphisms to Grassmannians

Just as base point free line bundles on a curve induce morphisms to projective space, globally generated vector bundles induce morphisms to Grassmannian varieties. We outline this constructions in somewhat greater detail, since it is not treated extensively in the literature.

Let E be a globally generated rank r bundle on C. By global generation, the sequence

$$0 \to H^0(C, E(-p)) \to H^0(C, E) \to E|_p \to 0$$

is exact for every $p \in C$, hence every fiber $E|_p$ is an r-dimensional quotient of $V = H^0(C, E)$. By associating

$$p \mapsto \left[V \to E|_p \to 0\right]$$

we get a map $\Phi_E \colon C \to G(V, r)$ to the Grassmannian of rank r quotients of V. In analogy with line bundles and maps to projective space we have:

Lemma 1.51. *Let E be a rank r vector bundle on C with $V = H^0(C, E)$. This data corresponds to an embedding $C \to G(V, r)$ if and only if*

a) V is base point free, i.e., $h^0(C, E(-p)) = h^0(C, E) - r$ for all $p \in C$,

b) V separates points, i.e., $h^0(C, E(-p-q)) < h^0(C, E) - r$ for all $p, q \in C$,

c) V separates tangent vectors, i.e., $h^0(C, E(-2p)) < h^0(C, E) - r$ for all $p \in C$.

We now compare the embedding given by $C \to G(V, r)$, followed by the Plücker embedding of $G(V, r)$, to the morphism given by the determinant of E. Since E is base point free, $L := \det(E) = \wedge^r E$ is as well. The induced map $\varphi_{|L|}$ is given by

$$C \to \mathbb{P}^*(H^0(C, L)), \quad p \mapsto \left[H^0(C, L) \to L|_p \to 0\right]$$

on closed points. There is also a map

$$\lambda\colon \wedge^r H^0(C, E) \to H^0(C, \wedge^r E), \quad s_1 \wedge \cdots \wedge s_r \mapsto [s_1 \wedge \cdots \wedge s_r]$$

which induces the rational map $\mathbb{P}^*(\lambda)\colon \mathbb{P}^*(H^0(C, L)) \to \mathbb{P}^*(\wedge^r V)$ by letting

$$\left[H^0(C, L) \to Q \to 0\right] \mapsto \left[\wedge^r V \to H^0(C, L) \to Q \to 0\right]$$

This rational map is defined on a quotient Q as long as the image of λ still surjects onto Q (which is true on an open subset of $\mathbb{P}^*(H^0(C, L))$). The Plücker embedding $G(V, r) \to \mathbb{P}^*(\wedge^r V)$ is given by

$$\left[V \to Q \to 0\right] \mapsto \left[\wedge^r V \to \wedge^r Q \to 0\right]$$

and hence we get a commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & G(V, r) \arrow[d] \\ \mathbb{P}^*(H^0(C, L)) \arrow[r] & \mathbb{P}^*(\wedge^r V) \end{tikzcd} \tag{1.3}$$

Note that if λ is surjective and the diagram is cartesian, then C is the transversal intersection of $G(V, r)$ and $\mathbb{P}^*(H^0(C, L))$.

1.11 Mukai bundles on curves in low genus

The general curve up to genus 5 can be described as a complete intersection in projective space. This is no longer true for curves of genus at least 6. However, if we allow more general homogeneous spaces instead of just projective space, curves of genus up to 9 are still complete intersections.

This path was first taken by S. Mukai. In the series of papers [Muk93; Muk95; Muk10] he exhibited the general curve C of genus $6 \leqslant g \leqslant 9$ as a (quadri)linear section of a Grassmannian variety X_g. More precisely, we have:

- $g = 6$: C is the intersection in $\mathbb{P}^9$ of a $G(2,5)$, a $\mathbb{P}^5$ and a quadric hypersurface if and only if $W^1_4(C)$ is finite.
- $g = 7$: C is the intersection in $\mathbb{P}^{15}$ of $OG(5,10)$, the orthogonal Grassmannian of a 10-dimensional vector space, and a $\mathbb{P}^6$ if and only if $W^1_4(C) = \emptyset$.
- $g = 8$: C is the intersection in $\mathbb{P}^{14}$ of a $G(2,6)$ and a $\mathbb{P}^7$ if and only if $W^2_7(C) = \emptyset$.
- $g = 9$: C is the intersection in $\mathbb{P}^{13}$ of $SpG(3,6)$, the Grassmannian of Lagrangian subspaces of a 6-dimensional vector space, and a $\mathbb{P}^8$ if and only if $W^1_5(C) = \emptyset$.

In each case the embedding of the curve C is induced by a unique stable vector bundle E_C with $\det E_C = \omega_C$. It has rank 2, 5, 2 and 3 for genus 6, 7, 8 and 9, respectively. These bundles have an unusually high amount of global sections.

Definition 1.52. We call E_C the *Mukai bundle* of C.

The geometry of general curves of genus $6 \leqslant g \leqslant 9$ seems to be mostly governed by their Mukai bundles. In the cases of genus 6 and 8, which are the most pertinent for this thesis, we sketch the construction of E_C. For more details, we refer the reader to [Muk93, Sections 3 and 5].

The central observation is that E_C is characterized by the following uniqueness properties:

Theorem 1.53 ([Muk93, §5])**.** *Let C be a curve of genus 6 which is neither trigonal nor a plane quintic. When F runs over all stable rank 2 bundles with canonical determinant on C, the maximum of $h^0(C,F)$ is equal to 5. Moreover, such vector bundles E_C on C with $h^0(C,E_C) = 5$ are unique up to isomorphism and generated by global sections.*

Theorem 1.54 ([Muk93, §3])**.** *Let C be a curve of genus 8 without a $\mathfrak{g}^2_7$. When F runs over all stable rank 2 bundles with canonical determinant on C, the maximum of $h^0(C,F)$ is equal to 6. Moreover, such vector bundles E_C on C with $h^0(C,E_C) = 6$ are unique up to isomorphism and generated by global sections.*

To actually construct the bundle E_C in question, one considers extensions of line bundles with Brill–Noether number $\rho(g, r, d) = 0$. By Theorem 1.33 there are finitely many of these line bundles on a general curve and one can show that the construction of E_C does not depend on the choices.

Lemma 1.55. *Let C be of genus 6 and $A \in W^1_4(C)$. Set $L = \omega_C \otimes A^{-1}$. The bundle E_C is given as the unique nontrivial extension of L by A with a 5-dimensional space of global sections.*

The main steps of the construction are as follows. Consider any extension

$$0 \to A \to F \to L \to 0$$

and the resulting exact sequence in cohomology:

$$0 \to H^0(C, A) \to H^0(C, F) \to H^0(C, L) \xrightarrow{\delta_F} H^1(C, A) \to \cdots$$

Then $h^0(C, F) \leqslant h^0(C, A) + h^0(C, L) = 5$ with equality if and only if $\delta_F = 0$. By Serre duality we have

$$\mathrm{Ext}^1(L, A) \cong H^1(C, A \otimes L^{-1}) \cong H^0(C, L^{\otimes 2})^\vee$$

while the boundary homomorphism δ_F lies in

$$\begin{aligned} \mathrm{Hom}(H^0(C, L), H^1(C, A)) &= \mathrm{Hom}(H^0(C, L), H^0(C, L)^\vee) \\ &= H^0(C, L)^\vee \otimes H^0(C, L)^\vee \end{aligned}$$

We have a map

$$\mathrm{Ext}^1(L, A) = H^0(C, L^{\otimes 2})^\vee \to H^0(C, L)^\vee \otimes H^0(C, L)^\vee, \tag{1.4}$$

given by $[F] \mapsto \delta_F$, which is dual to the multiplication map of sections

$$H^0(C, L) \otimes H^0(C, L) \to H^0(C, L^{\otimes 2})$$

It turns out that the cokernel of this map is $H^0(C, L^{\otimes 2}) / \mathrm{Sym}^2 H^0(C, L)$, which is 1-dimensional. Hence, up to scaling there is a unique nonzero element in the kernel of the map (1.4), i.e., a unique extension class $[E_C]$ with zero boundary homomorphism δ_{E_C}. It can be checked that this construction does not depend on the choice of A.

Remark 1.56. Let C a bielliptic curve of genus 6. Following Theorem 1.53, there still exists a stable rank two Mukai bundle E_C. However, the map $C \to G(2, 5)$ induced by E_C is not an embedding. Instead, we recover the double cover $C \to E$ where the elliptic curve E is embedded in $G(2, 5)$.

By an analogous construction in genus 8, we get the following result:

Lemma 1.57. *Let* C *be of genus* 8 *and* $A \in W^1_5(C)$. *Set* $L = \omega_C \otimes A^{-1}$. *The bundle* E_C *is given as the unique nontrivial extension of* L *by* A *with a* 6*-dimensional space of global sections.*

If C is a curve of genus 6 with $W^1_4(C)$ finite or a curve of genus 8 with $W^2_7(C) = \emptyset$, then the map λ in section 1.10 is indeed surjective. Additionally, in the genus 8 case, the diagram (1.3) is cartesian and hence C is the complete intersection of $G(6,2)$ and $\mathbb{P}^*(H^0(C, \omega_C))$.

1.12 The normal bundle of canonical curves

To an embedded curve $C \hookrightarrow \mathbb{P}^r$ with $L = \mathcal{O}_C(1)$ we can naturally attach two distinguished vector bundles. First there is the restricted cotangent bundle $\Omega_{\mathbb{P}^r}(1)|_C$ which by the Euler sequence coincides with the kernel bundle M_L. As discussed in section 1.9, a lot is known about the geometry of M_L. For instance, it governs the syzygies of the embedding of C, and a simple argument (like in [ES12]) shows that in most cases M_L is a stable vector bundle. In particular this is true for the canonical embedding, i.e., for $L = \omega_C$.

The second distinguished vector bundle is the normal bundle $\mathcal{N}_{C/\mathbb{P}^r}$ of the embedded curve. Of particular interest is the normal bundle $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ of the canonical embedding of C. As it turns out, the study of normal bundles is much more delicate than the study of the restricted cotangent bundles. In particular, not much is known about their stability properties. To discuss this, we will make use of the following term:

Definition 1.58. A vector bundle on a curve C is called *polystable* if it splits into the direct sum of stable bundles, all of the same slope.

Of course every stable bundle is polystable and polystable bundles are semistable.

Using the Euler exact sequence and the normal bundle exact sequence we calculate that for any canonically embedded curve

$$\det(\mathcal{N}_{C/\mathbb{P}^{g-1}}) = \omega_C^{\otimes(g+1)}, \quad \deg(\mathcal{N}_{C/\mathbb{P}^{g-1}}) = (g+1)(2g-2)$$

and we remark that $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ is of rank $g-2$.

We now consider the stability of the normal bundle of canonical curves in low genus. Every non-hyperelliptic genus 3 curve is canonically embedded as a quartic in $\mathbb{P}^2$, hence the normal bundle is a line bundle and therefore stable. For genus 4 and 5, the general canonical curve is the complete intersection of a cubic and a quadric, or three quadrics, respectively. Since the normal bundle of a complete intersection splits, we get

$$\mathcal{N}_{C/\mathbb{P}^{g-1}} = \begin{cases} \mathcal{O}_C(2) \oplus \mathcal{O}_C(3), & g = 4 \\ \mathcal{O}_C(2) \oplus \mathcal{O}_C(2) \oplus \mathcal{O}_C(2), & g = 5 \end{cases}$$

We see that for genus 4 the normal bundle is unstable while it is polystable (but not stable) for genus 5.

Moving forward to genus 6, there the general curve is tetragonal. The following result then implies that all canonical curves of genus 6 have unstable normal bundle:

Proposition 1.59 ([AFO16, Proposition 3.2]). *The normal bundle of a tetragonal canonical curve of genus $g \geqslant 6$ is unstable.*

However one of the main results of the same paper [AFO16] is that in genus 7 the normal bundle of the general curve is indeed stable:

Theorem 1.60 ([AFO16, Theorem 0.3]). *The normal bundle $\mathcal{N}_{C/\mathbb{P}^6}$ of every non-tetragonal canonical curve C of genus 7 is stable.*

The main fact used in the proof is the identification of the twist $\mathcal{N}^{\vee}_{C/\mathbb{P}^6}(2)$ of the conormal bundle with the Mukai bundle E_C discussed in section 1.11. Observe that the general curve of genus at least 7 is not tetragonal. Based on the previous considerations, the authors then make the following conjecture:

Conjecture 1.61 ([AFO16, Conjecture 0.4]). *The normal bundle of a general canonical curve of genus $g \geqslant 7$ is stable.*

In chapter 4 we will prove a similar result for genus 8 and go some way in establishing the equivalent in genus 9. Of course one would like a simple characterization of the locus in $\mathcal{M}_g$ of curves with (semi-)stable normal bundle. The results up to now suggest that a characterization in terms of the Clifford index or the gonality could be possible.

In another direction we can ask not for the canonical embedding but for *any* embedding of C of sufficiently high degree. Based on some heuristics one could conjecture that in this situation the normal bundles will always be stable.

Question 1.62. If a (general) curve of genus g is embedded by a complete linear system of degree $d \gg 0$, is its normal bundle stable?

The only result so far in that direction is given by L. Ein and R. Lazarsfeld in [EL92] where they prove polystability for normal bundles of elliptic curves of high degree.

1.13 Outline of results

In Chapter 2 we investigate the birational geometry of $\overline{\mathcal{R}}_{15,2}$. As claimed in Theorem 1.46, this space turns out to be of general type:

Theorem 1.63. *The moduli space $\overline{\mathcal{R}}_{15,2}$ is of general type.*

We achieve this result by following the strategy outlined in section 1.6. This means that we describe an effective divisor $\mathcal{D}_{15}$, calculate the class of its closure in terms of the standard basis of $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}_{15,2})$ and show that this makes the canonical class of $\overline{\mathcal{R}}_{15,2}$ big. $\mathcal{D}_{15}$ is constructed as the locus where a certain map between vector bundles over $\mathcal{R}_{15,2}$ drops rank.

We continue the study of level curves in Chapter 3. On a general curve of genus 6 and 8 we have the uniquely defined rank 2 Mukai bundle E_C, which was discussed in section 1.11. For every level ℓ, we study the theta divisor of twists of E_C on $\mathcal{R}_{6,\ell}$ and $\mathcal{R}_{8,\ell}$. After investigating the geometry of these twists in detail, we calculate the classes of the theta divisors. The instance in genus 8 and level 3 is then used to prove that $\mathcal{R}_{8,3}$ is of general type:

Theorem 1.64. *$\overline{\mathcal{R}}_{8,3}$ is of general type.*

Finally, in Chapter 4 we start by studying the stability of the normal bundle of canonical genus 8 curves.

Theorem 1.65. *The normal bundle of a canonical curve C of genus 8 is stable if and only if the curve does not have a $\mathfrak{g}^2_7$. Furthermore, it is polystable if and only if the curve is not tetragonal.*

To prove this result, we consider the embedding $C \to G(2,6)$ induced by the Mukai bundle of C. Then the normal bundle of C in its canonical embedding is a restriction of the normal bundle of $G(2,6)$, embedded by the Plücker embedding in $\mathbb{P}^{14}$. This description allows us to provide explicit calculations proving the stability.

For canonical genus 9 curves the situation seems to be considerably more complicated, although we still have an embedding $C \to \mathrm{SpG}(3,6)$ given by the Mukai bundle of C. Using this, we are at least able to prove stability with respect to subbundles of low ranks.

Theorem 1.66. *The stability degrees of the twisted conormal bundle $\mathcal{N} = \mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$ of a general canonical genus 9 curve are bounded as follows:*

$$\begin{aligned} s_1(\mathcal{N}) &= 36 \\ s_2(\mathcal{N}) &\geqslant 9 \\ s_3(\mathcal{N}) &\geqslant -18 \\ s_4(\mathcal{N}) &\geqslant -38 \\ s_5(\mathcal{N}) &\geqslant -44 \\ s_6(\mathcal{N}) &\geqslant -8 \end{aligned}$$

In particular, $\mathcal{N}$ is stable with respect to subbundles of ranks 1 and 2. Additionally, if $\mathcal{N}$ has no $\mathfrak{g}^2_8$ quotient line bundle, then $s_6(\mathcal{N}) \geqslant 6$.

In an afterword, we also offer some more evidence for conjecture 1.61, that a general canonical curve of every genus $g \geqslant 7$ has stable normal bundle.

This thesis contains material already published on the arXiv. Chapter 2 is essentially the article [Bru16a], stripped of the introduction. The material in Chapter 3 is, apart from minor adjustments, equal to the preprint [Bru16b].

CHAPTER 2

The Kodaira dimension of $\overline{\mathcal{R}}_{15,2}$

The goal of this chapter is to prove the following theorem:

Theorem 2.1. *The moduli space $\overline{\mathcal{R}}_{15,2}$ is of general type.*

As outlined in section 1.8, this completes the birational classification of $\overline{\mathcal{R}}_{g,2}$ for $g \geqslant 14$. We briefly describe the strategy of the proof. Methodically we proceed along the lines of section 1.6. In order to show that the canonical class of $\overline{\mathcal{R}}_{15,2}$ is big, we construct an effective divisor $\mathcal{D}_{15}$ such that $K_{\overline{\mathcal{R}}_{15,2}}$ can be written as a positive linear combination of the Hodge class, the class of $\overline{\mathcal{D}}_{15}$ and other effective divisor classes.

To motivate the construction of $\mathcal{D}_{15}$, consider first the case of genus 6. A general curve $[C] \in \mathcal{M}_6$ possesses a finite number of complete $\mathfrak{g}^2_6$. Any such $L \in W^2_6(C)$ induces a birational map to a plane sextic curve Γ with 4 nodes. If there is a plane conic Q totally tangent to Γ, i.e., $Q \cdot \Gamma = 2D$ where D is effective of degree 6, then $\eta = \mathcal{O}_\Gamma(-1) \otimes \mathcal{O}_\Gamma(D)$ is 2-torsion.

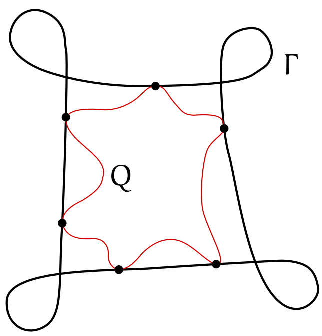

Figure 2.1: A 4-nodal sextic plane curve Γ with a totally tangent conic Q.

The existence of such a totally tangent conic is equivalent to the failure of the map

$$\operatorname{Sym}^2 H^0(C, L \otimes \eta) \to \frac{H^0(C, L^{\otimes 2})}{\operatorname{Sym}^2 H^0(C, L)}$$

to be injective. It turns out that the closure of the locus of pairs $[C, \eta] \in \mathcal{R}_{6,2}$ where this injectivity fails for some $L \in W^2_6(C)$ is a divisor, i.e., the condition to possess a totally tangent conic to a plane sextic model gives a divisorial condition on $\mathcal{R}_{6,2}$. This divisor can also be identified with the closure of the ramification divisor of the Prym map $\mathcal{R}_{6,2} \to \mathcal{A}_5$. For details, see [FGSV14].

We generalize this condition and adapt it to genus 15. A general genus 15 curve C carries a finite number of complete $\mathfrak{g}^4_{16}$ linear series. For any such $L \in W^4_{16}(C)$ we can consider the multiplication map

$$\mu_{[C,L]} \colon \operatorname{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

The vector spaces on the left and right hand side are of dimensions 15 and 18, respectively, and the map is injective for the general pair $[C, L]$. We can make use of a torsion bundle η to get the remaining three sections:

$$\mu_{[C,\eta,L]} \colon \operatorname{Sym}^2 H^0(C, L) \oplus \operatorname{Sym}^2 H^0(C, L \otimes \eta) \to H^0(C, L^{\otimes 2}) \tag{2.1}$$

We consider the locus of Prym curves carrying a $\mathfrak{g}^4_{16}$ such that this map fails to be an isomorphism. Unlike in genus 6, such curves are not directly characterized by having a totally tangent quadric hypersurface, although on those that have, the map (2.1) certainly fails to be injective.

It turns out that $\mu_{[C,\eta,L]}$ is bijective for all L on the general pair $[C, \eta] \in \mathcal{R}_{15,2}$ and the failure locus is in codimension one. We may therefore consider the divisor

$$\mathcal{D}_{15} = \left\{ [C, \eta] \in \mathcal{R}_{15,2} \;\middle|\; \exists L \in W^4_{16}(C) \colon \mu_{[C,\eta,L]} \text{ not an isomorphism} \right\}$$

In order to show that (2.1) is indeed bijective for all η and L on a general curve C, we first construct in section 2.2.1 a single example, using a curve that carries a theta characteristic with a large number of sections. Afterwards we prove that the moduli space $\mathfrak{G}^{4,(2)}_{16}$ of triples $[C, \eta, L]$ is irreducible, allowing us to specialize the general triple to the constructed example. More generally, we obtain the following result:

Proposition 2.2. *Fix $g \geqslant 3$ and let $\ell \geqslant 2$ be a prime. Assume that the Brill–Noether number $\rho(g, r, d) = 0$. If either $r \leqslant 2$ or $g - d + r - 1 \leqslant 2$ then $\mathfrak{G}^{r,(\ell)}_d$, the moduli space of triples $[C, \eta, L]$ with $[C, \eta] \in \mathcal{R}_{g,\ell}$ and $L \in W^r_d(C)$, is irreducible.*

Taking the closure $\overline{\mathcal{D}}_{15}$ of $\mathcal{D}_{15}$ in an appropriate partial compactification $\overline{\mathcal{R}}^0_{15}$ of $\mathcal{R}_{15,2}$, we can calculate the class of $\overline{\mathcal{D}}_{15}$ using a determinantal description coming from globalizing the map (2.1) to a morphism of vector bundles over $\mathfrak{G}^{4,(2)}_{16}$.

Theorem 2.3. *The locus $\overline{\mathcal{D}}_{15}$ is a divisor in $\overline{\mathcal{R}}^0_{15}$ and we have the expression*

$$[\overline{\mathcal{D}}_{15}] + \mathsf{E} \equiv 31020\left(\frac{3127}{470}\lambda - (\delta_0' + \delta_0'') - \frac{3487}{1880}\delta_0^{\mathrm{ram}}\right)$$

where E is an effective class on $\overline{\mathcal{R}}^0_{15}$.

A suitable positive linear combination of $\overline{\mathcal{D}}_{15}$ and another divisor $\overline{\mathcal{D}}_{15:2}$, which was described in [FL10], then shows that the canonical class of $\overline{\mathcal{R}}_{15,2}$ is big.

To be able to calculate the class of $\overline{\mathcal{D}}_{15}$, various technical difficulties have to be overcome. In section 2.1 we closely follow the set-up of [Far09] and [FL10] to construct partial compactifications of suitable open subsets of $\mathcal{M}_g$ and $\mathcal{R}_{g,2}$ and to extend the moduli stacks of linear series there. We also make use of Theorem 1.20 showing that all pluricanonical forms defined on the smooth part of $\overline{\mathcal{R}}_{g,2}$ extend to any resolution of singularities.

2.1 Setting up the moduli spaces

2.1.1 The universal Prym curve

Since by using Lemma 1.26 we can restrict our attention to the boundary divisors Δ_0', Δ_0'' and Δ_0^{ram}, we partially compactify $\mathcal{M}_g$ by adding the open sublocus $\widetilde{\Delta}_0 \subset \Delta_0$ of one-nodal irreducible curves. Set

$$\widetilde{\mathcal{M}}_g = \mathcal{M}_g \cup \widetilde{\Delta}_0$$

and let $\widetilde{\mathcal{R}}_g = \pi^{-1}(\widetilde{\mathcal{M}}_g)$. We also set

$$\mathcal{Z} = \widetilde{\mathsf{R}}_g \times_{\widetilde{\mathsf{M}}_g} \widetilde{\mathsf{M}}_{g,1}$$

This is not yet the universal Prym curve over $\widetilde{\mathsf{R}}_g$, since the points on exceptional components of curves in Δ_0^{ram} are not present. We have to blow up $\mathcal{Z}$ along the locus V of points

$$(X, \eta_X, p = q) \in \Delta_0^{\mathrm{ram}} \times_{\widetilde{\mathsf{M}}_g} \widetilde{\mathsf{M}}_{g,1}, \quad X = C \cup_{\{p,q\}} E \to C/p \sim q, \quad \eta_E = \mathcal{O}_E(1)$$

Set $\mathcal{X}_g = \mathrm{Bl}_V(\mathcal{Z})$ and let $f\colon \mathcal{X}_g \to \widetilde{\mathsf{R}}_g$ be the induced universal family of Prym curves. The family $\mathcal{X}_g$ comes equipped with a Poincaré bundle $\mathscr{P}$ such that $\mathscr{P}|_{f^{-1}([X,\eta,\beta])} = \eta$. We need the following result from [FL10, Proposition 1.6]:

Lemma 2.4. *In* $\mathrm{Pic}(\widetilde{\mathcal{R}}_g)$ *we have*

$$f_*(c_1^2(\mathscr{P})) = -\delta_0^{\mathrm{ram}}/2$$

and

$$f_*(c_1(\mathscr{P})c_1(\omega_X)) = 0$$

2.1.2 Moduli spaces of linear series over the Prym moduli space

To compute the classes of divisors on $\overline{\mathcal{R}}_{g,2}$, a viable method is to give them a determinantal description, i.e., exhibit them as degeneracy loci of morphisms of vector bundles. To obtain these vector bundles, we will consider the stack $\mathfrak{G}^{r,(2)}_d$ parametrizing triples $[C,\eta,L]$ where $[C,\eta] \in \mathcal{R}_{g,2}$ and $L \in G^r_d(C)$. Note that in the case $\rho(g,r,d) = 0$ in which we are interested, the forgetful map $\mathfrak{G}^{r,(2)}_d \to \mathsf{R}_{g,2}$ is a generically finite cover of degree

$$N = g!\,\frac{1!\,2!\cdots r!}{(g-d+r)!\cdots(g-d+2r)!}$$

We want to first restrict this construction to an open subset of $\mathcal{R}_{g,2}$ such that various push-forwards of the Poincaré bundles on the universal curve are indeed vector bundles on $\mathfrak{G}^{r,(2)}_d$. Then we shall extend the stack over a suitable partial compactification to be able to also determine the behavior on the boundary.

Let M^0_g be the open substack of M_g classifying curves C with $W^{r+1}_d(C) = \emptyset$ and $W^r_{d-1}(C) = \emptyset$. A general such curve indeed has a finite amount of $\mathfrak{g}^r_d$ linear series and all of them are very ample. Observe that both

$$\rho(g,r+1,d) = -(g-d+2(r+1)) \leqslant -2, \quad \rho(g,r,d-1) = -(r+1) \leqslant -2$$

so the codimension of the complement of $\mathcal{M}^0_g$ in $\mathcal{M}_g$ is at least 2 (see the results in section 1.7). Therefore, restricting to $\mathcal{M}^0_g$ does not change divisor class calculations.

To partially compactify M^0_g, add the locus Δ^0_0 of Brill–Noether general irreducible one-nodal curves, i.e., $[C/p \sim q]$ with $[C] \in \mathcal{M}_{g-1}$ satisfying the Brill–Noether theorem. Denote by $\overline{\mathsf{M}}^0_g = \mathsf{M}^0_g \cup \Delta^0_0$ the resulting partial compactification. Over $\overline{\mathsf{M}}^0_g$ we consider the stack of pairs $[C,L]$ where $L \in G^r_d(C)$. We denote this stack by $\overline{\mathfrak{G}}^r_d$. Pulling back the universal curve $\overline{\mathsf{M}}^0_{g,1}$ to $\overline{\mathfrak{G}}^r_d$, we get a universal family

$$f^r_d\colon \overline{\mathfrak{C}}^r_d = \overline{\mathfrak{G}}^r_d \times_{\overline{\mathsf{M}}^0_g} \overline{\mathsf{M}}^0_{g,1} \to \overline{\mathfrak{G}}^r_d$$

and we choose a Poincaré bundle on it, i.e., an $\mathscr{L} \in \mathrm{Pic}(\overline{\mathfrak{C}}^r_d)$ such that for every $[C,L] \in \overline{\mathfrak{G}}^r_d$ we have $\mathscr{L}|_{(f^r_d)^{-1}([C,L])} = L$.

We are now ready to pull these constructions back to Prym curves. Let $\overline{\mathsf{R}}^0_g = \pi^{-1}(\overline{\mathsf{M}}^0_g)$ and

$$\sigma\colon \overline{\mathfrak{G}}^{r,(2)}_d = \overline{\mathfrak{G}}^r_d \times_{\overline{\mathsf{M}}^0_g} \overline{\mathsf{R}}^0_g \to \overline{\mathsf{R}}^0_g$$

be the stack parametrizing triples $[C,\eta,L]$ for $[C,\eta] \in \overline{\mathsf{R}}^0_g$ and $L \in W^r_d(C)$. We also have the universal curve

$$\chi\colon \overline{\mathfrak{C}}^{r,(2)}_d = \mathcal{X}_g \times_{\overline{\mathsf{R}}^0_g} \overline{\mathfrak{G}}^{r,(2)}_d \to \overline{\mathfrak{G}}^{r,(2)}_d$$

By pulling back from $\overline{\mathsf{R}}_g^0$ and $\overline{\mathfrak{G}}_d^{r,(2)}$, respectively, this comes equipped with two Poincaré bundles $\mathscr{P}$ and $\mathscr{L}$. We can also use σ to pull back the boundary classes Δ_0', Δ_0'' and Δ_0^{ram} from $\overline{\mathsf{R}}_g^0$ to $\overline{\mathfrak{G}}_d^{r,(2)}$. Slightly abusing notation, the pullbacks will be denoted by the same symbols.

2.2 A new divisor on $\overline{\mathcal{R}}_{15,2}$

As before, we denote by $\chi\colon \overline{\mathfrak{C}}_{16}^{4,(2)} \to \overline{\mathfrak{G}}_{16}^{4,(2)}$ the universal curve and let $\mathscr{L}$ be a Poincaré bundle on $\overline{\mathfrak{C}}_{16}^{4,(2)}$. Furthermore, we let ω_χ be the relative dualizing sheaf of χ and $\sigma\colon \overline{\mathfrak{G}}_{16}^{4,(2)} \to \overline{\mathcal{R}}_{15}^0$ be the generically finite cover of degree 6006.

By construction of our moduli stacks and Grauert's theorem, the pushforwards of $\mathscr{L}$ and $\mathscr{L}^{\otimes 2}$ by χ are vector bundles on $\overline{\mathfrak{G}}_{16}^{4,(2)}$ of ranks 5 and 18, respectively. The sheaf $\chi_*(\mathscr{L} \otimes \mathscr{P})$ is possibly not a vector bundle, but at least it is torsion-free. By excluding the subvariety (of codimension at least two) where it fails to be locally free we can assume it is in fact a vector bundle of rank 2. Divisor class calculations will not be affected.

We may then consider the following morphism of vector bundles of the same rank:

$$\phi\colon\ \mathrm{Sym}^2 \chi_*(\mathscr{L}) \oplus \mathrm{Sym}^2 \chi_*(\mathscr{L} \otimes \mathscr{P}) \to \chi_*(\mathscr{L}^{\otimes 2})$$

On the fiber over the class of a triple $[C, \eta, L]$ it is given by the multiplication map of sections

$$\mu_{[C,\eta,L]}\colon\ \mathrm{Sym}^2 H^0(C, L) \oplus \mathrm{Sym}^2 H^0(C, L \otimes \eta) \to H^0(C, L^{\otimes 2}) \qquad (2.2)$$

The closure of the locus

$$\mathcal{D}_{15} = \left\{ [C, \eta] \in \mathcal{R}_{15,2} \;\middle|\; \exists L \in W_{16}^4(C)\colon \mu_{[C,\eta,L]} \text{ not an isomorphism} \right\}$$

therefore has a determinantal description as the pushforward of the first degeneracy locus of the map ϕ. Its expected codimension is one and we obtain a virtual divisor. Note that while the vector bundles involved in defining ϕ clearly depend on the choice of the Poincaré bundle $\mathscr{L}$, the resulting morphism ϕ does not (cf. the remark before [Far09, Theorem 2.1]).

2.2.1 Proof of divisoriality of $\mathcal{D}_{15}$

We now prove that $\overline{\mathcal{D}}_{15}$ is a genuine divisor, that is $\mu_{[C,\eta,L]}$ is an isomorphism for every $L \in W_{16}^4(C)$ on the general Prym curve $[C, \eta]$. We will prove in section 2.2.2 that $\mathfrak{G}_{16}^{4,(2)}$ over the whole space $\mathcal{R}_{15,2}$ is irreducible. Hence it will be enough to exhibit a single smooth curve C and two line bundles $L \in W_{16}^4(C)$ and $\eta \in \mathrm{Pic}^0(C)[2]$ such that the multiplication map (2.2) is bijective. We can then specialize the general element of $\mathfrak{G}^{4,(2)}_{16}$ to this particular example and conclude by semicontinuity.

We start with a smooth nonhyperelliptic curve $C \in \mathcal{M}_{15}$ possessing a theta characteristic ϑ with exactly 5 sections, i.e., $|\vartheta| \in G^4_{14}(C)$ and $\vartheta^{\otimes 2} \cong \omega_C$. In order to construct an L such that $\mu_{[C,\eta,L]}$ is bijective, C should in fact be half-canonically embedded by ϑ such that the image does not lie on any quadric hypersurface in $\mathbb{P}^4$.

Explicit models of such curves can be obtained as hyperplane sections of smooth canonical surfaces $S \subseteq \mathbb{P}^5$ with $p_g = 6$ and $K_S^2 = 14$. To construct such a surface, one can employ the method described by F. Catanese in [Cat97].

Lemma 2.5. *There exists a smooth projective surface S of general type with invariants* $(K_S^2, p_g, q) = (14, 6, 0)$*, canonically embedded in* $\mathbb{P}^5$*, not lying on any quadric hypersurface.*

Proof. The surfaces S arise from Pfaffian resolutions of the ideal sheaf $\mathcal{I}_S$

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-7) \to \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 7} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 7} \xrightarrow{p} \mathcal{I}_S \to 0 \tag{2.3}$$

where α is a 7×7 antisymmetric matrix with linear entries and p is the map given by the Pfaffians of 6×6 principal submatrices of α.

Using the projective resolution (2.3) and Serre duality for Ext sheaves, we see that S is canonically embedded. We also see that S is a regular surface (i.e. $q = 0$) and $p_g = 6$ which combines to give $\chi(\mathcal{O}_S) = 7$. Again using (2.3), the Hilbert polynomial of $\mathcal{O}_S$ is $P_S(t) = 7t^2 - 7t + 7$ which tells us $\deg(S) = 14$ and because S is canonically embedded we have $K_S^2 = 14$. ∎

A general hyperplane section $C = H \cap S$ of S has, by the adjunction formula,

$$\omega_C \cong (\mathcal{O}_S(1) \otimes \omega_S)|_C \cong \omega_S^{\otimes 2}|_C, \quad 2g - 2 = 2K_S \cdot K_S = 28$$

so $C \hookrightarrow \mathbb{P}^4$ is half-canonically embedded of degree 14 and genus 15. Using the exact sequence

$$0 \to \mathcal{I}_S(2) \to \mathcal{O}_{\mathbb{P}^5}(2) \to \mathcal{O}_S(2) \to 0$$

and $h^0(S, \omega_S^{\otimes 2}) = 21$ by Riemann–Roch, we get $H^0(\mathbb{P}^5, \mathcal{I}_S(2)) = 0$, so S does not lie on a quadric hypersurface of $\mathbb{P}^5$. The same then applies for C in $\mathbb{P}^4$. A moduli count shows that hyperplane sections of such S form a 32-dimensional family.

Remark 2.6. This is not the only way in which such curves arise. A. Iliev and D. Markushevich ([IM00]) also obtain a 32-dimensional family (i.e. an irreducible component of the expected dimension of the locus $\mathcal{T}^4_{15}$ of curves of genus 15 having a theta-characteristic with 5 independent global sections) as vanishing loci of sections of rank 2 ACM bundles on quartic 3-folds in $\mathbb{P}^4$.

Lemma 2.7. *For a half-canonically embedded curve* C *in* $\mathbb{P}^4$ *not lying on a quadric hypersurface, the multiplication map* $\mu_{[C,\eta,L]}$ *is bijective.*

Proof. Set $\vartheta = \mathcal{O}_C(1)$. Of course $\mathcal{O}_C(2) = \omega_C$. The fact that C does not lie on a quadric hypersurface is equivalent to the bijectivity of the multiplication map

$$\mu_\vartheta \colon \operatorname{Sym}^2 H^0(C,\vartheta) \to H^0(C,\omega_C)$$

We now choose any closed point $x \in C$. Using that ϑ is very ample we get

$$h^0(C,\vartheta(-2x)) = h^0(C,\vartheta) - 2$$

By Serre duality this implies $h^0(C,\vartheta(2x)) = h^0(C,\vartheta)$. Let $L = \vartheta(2x)$, so L is a complete $\mathfrak{g}^4_{16}$ and $2x$ is contained in the base locus of L. In particular, we have $H^0(C,L) \cong H^0(C,\vartheta)$ and $|L| = |\vartheta| + 2x$. Taking symmetric powers, we get

$$\operatorname{Sym}^2 H^0(C,L) \cong \operatorname{Sym}^2 H^0(C,\vartheta) \cong H^0(C,\omega_C)$$

The space $H^0(C,L^{\otimes 2})$ is 18-dimensional and decomposes via the inclusion $H^0(C,\vartheta^{\otimes 2}) \hookrightarrow H^0(C,L^{\otimes 2})$ as

$$H^0(C,L^{\otimes 2}) \cong H^0(C,\omega_C) \oplus V \cong \operatorname{Sym}^2 H^0(C,L) \oplus V$$

where $\dim V = 3$. The sections in $\operatorname{Sym}^2 H^0(C,L)$ vanish to orders at least 4 at x. By Riemann–Roch, the space $H^0(C,L^{\otimes 2})$ does contain sections vanishing to orders 0, 1 and 2 at x. By the previous analysis, they must span V.

Choose a two-torsion bundle $\eta \in \operatorname{Pic}^0(C)[2]$ such that $H^0(C,\vartheta\otimes\eta) = 0$. Since $\operatorname{Pic}^0(C)[2]$ acts transitively on the theta-characteristics, such an η always exists by a result of Mumford ([Mum66]). Then we have

$$h^0(C,L\otimes\eta) = h^0(C,\vartheta(2x)\otimes\eta) \leqslant h^0(C,\vartheta\otimes\eta) + 2 = 2$$

By Riemann–Roch we must in fact have $h^0(C,L\otimes\eta) = 2$. By construction,

$$H^0(C,(L\otimes\eta)(-2x)) = H^0(C,\vartheta\otimes\eta) = 0,$$

so the two sections of $L\otimes\eta$ vanish to orders 0 and 1 at x. We conclude that the map

$$\operatorname{Sym}^2 H^0(C,L\otimes\eta) \to H^0(C,L^{\otimes 2})$$

is injective and its image is precisely V. ∎

2.2.2 Irreducibility of some spaces of linear series

We now want to prove the irreducibility of $\mathfrak{G}^{4,(2)}_{16}$, i.e., the moduli space of triples $[C,\eta,L]$ where $[C,\eta] \in \mathcal{R}_{15,2}$ and $L \in W^4_{16}(C)$. This will show that for the general triple $[C,\eta,L]$ the map $\mu_{[C,\eta,L]}$ is an isomorphism. Notice that the pair $[C,L]$ constructed in section 2.2.1 is *not* Petri general, so we need more than the existence of a unique component of $\mathfrak{G}^{4,(2)}_{16}$ dominating $\mathcal{M}_{15}$. Nonetheless, this fact is what we are going to establish first in greater generality:

Proposition 2.8. *Let $g \geqslant 3$ and $\ell \geqslant 2$ be a prime. Assume $\rho(g,r,d) = 0$. Then there is a unique irreducible component of $\mathfrak{G}^{r,(\ell)}_d$ dominating $\mathcal{M}_g$, i.e., containing the Petri general triple $[C, \eta, L]$.*

Proof. If $r = g-1$, the only $\mathfrak{g}^r_d$ on a curve is the canonical bundle, so the space $\mathfrak{G}^{r,(\ell)}_d \cong \mathcal{R}_{g,\ell}$ is irreducible. Otherwise set $k = g - d + r + 1 \geqslant 3$. We recall that the locus of Petri general pairs $[C, L]$ is a connected smooth open subset U of one irreducible component of $\mathfrak{G}^r_d$ (see section 1.7). The restriction of $\mathfrak{G}^{r,(\ell)}_d$ to the preimage $U^{(\ell)}$ of U is smooth, so in order to show $U^{(\ell)}$ is irreducible we only have to show it is connected.

Take a general k-gonal curve $[D, A]$. We then have $h^0(D, A^{\otimes j}) = j + 1$ for all $j \leqslant r+1$ (see [Bal89]). So there is a rational map

$$\Psi\colon \mathfrak{G}^{1,(\ell)}_k \dashrightarrow \mathfrak{G}^{r,(\ell)}_d$$

defined by $[D, \eta, A] \mapsto [D, \eta, A^{\otimes r}]$. We claim $A^{\otimes r}$ is Petri general, i.e., the map

$$\mu_{A^{\otimes r}}\colon H^0(D, A^{\otimes r}) \otimes H^0(D, \omega_D \otimes A^{\otimes(-r)}) \to H^0(D, \omega_D)$$

is injective. The aforementioned result of Ballico implies

$$h^0(D, \omega_D \otimes A^{\otimes(-j)}) = (k-1)(r+1-j)$$

for all $j \leqslant r+1$. Note also that $g = (k-1)(r+1)$. By counting dimensions we find that $\mu_{A^{\otimes r}}$ is injective if and only if it is surjective.

We write down the beginning of the long exact sequence coming from the base point free pencil trick:

$$0 \to H^0(\omega_D \otimes A^{\otimes(-j-1)}) \to H^0(A) \otimes H^0(\omega_D \otimes A^{\otimes(-j)}) \to H^0(\omega_D \otimes A^{\otimes(-j+1)})$$

Comparing dimensions we find that the map on the right is surjective for all $j \leqslant r$. Now we have to observe that $h^0(D, A^{\otimes r}) = r + 1$ is equivalent to $H^0(D, A^{\otimes r}) \cong \operatorname{Sym}^r H^0(D, A)$. The chain of surjective maps

$$\begin{aligned} H^0(A)^{\otimes r} \otimes H^0(\omega_D \otimes A^{\otimes(-r)}) \twoheadrightarrow H^0(A)^{\otimes(r-1)} \otimes H^0(\omega_D \otimes A^{\otimes(-r+1)}) \twoheadrightarrow \cdots \\ \cdots \twoheadrightarrow H^0(A) \otimes H^0(\omega_D \otimes A^{-1}) \end{aligned}$$

then implies that the Petri map

$$\mu_{A^{\otimes r}}\colon \operatorname{Sym}^r H^0(D, A) \otimes H^0(D, \omega_D \otimes A^{\otimes(-r)}) \to H^0(D, \omega_D)$$

is surjective as well. So $[D, \eta, A^{\otimes r}]$ lies in $U^{(\ell)}$.

In [BF86] it is shown that the twisted Hurwitz space $\mathfrak{G}^{1,(\ell)}_k$ is irreducible for $k \geqslant 3$. Hence Ψ maps to the smooth locus of a unique component Z of $\mathfrak{G}^{r,(\ell)}_d$ and its image is an irreducible subset consisting generically of Petri general curves. Since the image is closed under monodromy of ℓ-torsion, $U^{(\ell)}$ must be connected. ∎

We will employ this result to prove the irreducibility of $\mathfrak{G}^{r,(\ell)}_d$ under special circumstances:

Corollary 2.9. *Let* $g \geqslant 3$*, and let* $\ell \geqslant 2$ *be a prime. Assume that* $\rho(g,r,d) = 0$*. If* $r \leqslant 2$ *or* $r' = g - d + r - 1 \leqslant 2$ *then* $\mathfrak{G}^{r,(\ell)}_d$ *is irreducible.*

Proof. Note that the Serre dual of a $\mathfrak{g}^r_d$ is a $\mathfrak{g}^{r'}_{2g-2-d}$, so the space $\mathfrak{G}^{r,(\ell)}_d$ is irreducible if and only if $\mathfrak{G}^{r',(\ell)}_{2g-2-d}$ is. As mentioned above, if $r = 0$ or, equivalently, $r' = g - 1$, the unique $\mathfrak{g}^r_d$ on a curve is its canonical bundle, so $\mathfrak{G}^{r,(\ell)}_d \cong \mathcal{R}_{g,\ell}$ is irreducible. The case $r = 1$ is just the aforementioned result [BF86] by Biggers and Fried about the irreducibility of Hurwitz spaces.

In the remaining case $r = 2$ a general $\mathfrak{g}^2_d$ maps C birationally to a nodal curve in $\mathbb{P}^2$. Thus we get a dominant rational map

$$V^{d,g} \dashrightarrow \mathfrak{G}^2_d$$

from the Severi variety $V^{d,g}$ of irreducible plane curves of degree d and arithmetic genus g. The Severi varieties are irreducible, as proven in [Har84], so $\mathfrak{G}^2_d$ is irreducible as well.

Étale maps preserve dimension, so all components of $\mathfrak{G}^{2,(\ell)}_d$ have dimension $3g - 3 + \rho(g,r,d) = 3g - 3$. Each component is generically smooth, which implies that the general element has injective Petri map. But by Proposition 2.8 there is only one such component. ■

In particular, $\mathfrak{G}^{4,(2)}_{16}$ is irreducible. We may therefore specialize a general triple $[C,\eta,L] \in \mathfrak{G}^{4,(2)}_{16}$ to the previously constructed explicit example. This proves that the locus $\overline{\mathcal{D}}_{15}$ is a genuine divisor. We proceed to calculate its class.

2.2.3 Calculation of the divisor class

Recall that we are considering the morphism

$$\phi\colon \operatorname{Sym}^2 \chi_*(\mathscr{L}) \oplus \operatorname{Sym}^2 \chi_*(\mathscr{L} \otimes \mathscr{P}) \to \chi_*(\mathscr{L}^{\otimes 2})$$

between vector bundles of the same rank. To calculate the Chern classes of these bundles we will employ Grothendieck–Riemann–Roch. For this we study the contribution coming from $R^1\chi_*(\mathscr{L} \otimes \mathscr{P})$.

Lemma 2.10. *Let* $[C,\eta] \in \Delta_0''$ *be general and* $L \in W^4_{16}(C)$*. Then* $h^0(C, L \otimes \eta) = 4$*.*

Proof. Let $\nu\colon \widetilde{C} \to C$ be the normalization of C and x be the node. Then $\nu^*\eta = \mathcal{O}_{\widetilde{C}}$ and $\nu^*L \in W^4_{16}(\widetilde{C})$, since $\widetilde{C}$ is Brill–Noether general. From the exact sequence

$$0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_{\widetilde{C}} \xrightarrow{e} \mathbb{C}_x \to 0$$

we get

$$0 \to L \otimes \eta \to \nu_* \nu^* L \xrightarrow{e'} L \otimes \eta|_x \to 0$$

and by the long exact sequence in cohomology we obtain

$$0 \to H^0(C, L \otimes \eta) \to H^0(\widetilde{C}, \nu^* L) \xrightarrow{H^0(e')} \mathbb{C}$$

Now $H^0(e)$ is the zero map, hence $H^0(e')$ must be nonzero and we get

$$h^0(C, L \otimes \eta) = h^0(\widetilde{C}, \nu^* L) - 1 = 4 \qquad \blacksquare$$

This implies that the dimension of $h^0(C, L \otimes \eta)$ jumps by two on the boundary component Δ_0''. Hence $R^1\chi_*(\mathscr{L} \otimes \mathscr{P})$ is supported at least on Δ_0'', and there it is of rank 2.

Remark 2.11. In fact Δ_0'' seems to be the only divisor where $R^1\chi_*(\mathscr{L} \otimes \mathscr{P})$ is supported. Since a proof of this would take long, and is not strictly necessary to achieve the goal of the thesis, we do not assume this fact here and will discuss it in greater generality in future work.

Denote $\mathfrak{d} = c_1(R^1\chi_*(\mathscr{L} \otimes \mathscr{P}))$.

Proposition 2.12. *The class of the degeneracy locus $Z_1(\phi)$ in $\overline{\mathcal{R}}_{15}^0$ is*

$$[\overline{\mathcal{D}}_{15}]^{\text{virt}} \equiv 31020 \left(\frac{3127}{470}\lambda - (\delta_0' + \delta_0'') - \frac{3487}{1880}\delta_0^{\text{ram}} \right) - 3\sigma_*(\mathfrak{d})$$

and $[\overline{\mathcal{D}}_{15}]^{\text{virt}} - n[\overline{\mathcal{D}}_{15}]$ is an effective class entirely supported on the boundary for some $n \geqslant 1$.

Proof. We introduce the following classes in $A^1(\overline{\mathfrak{G}}_{16}^{4,(2)})$:

$$\mathfrak{a} = \chi_*(c_1^2(\mathscr{L})), \quad \mathfrak{b} = \chi_*(c_1(\mathscr{L}) \cdot c_1(\omega_\chi)), \quad \mathfrak{c} = c_1(\chi_*(\mathscr{L}))$$

By Porteous' formula, the class of the first degeneracy locus $Z_1(\phi)$ of ϕ is given by

$$Z_1(\phi) = c_1(\chi_*(\mathscr{L}^{\otimes 2})) - c_1(\operatorname{Sym}^2 \chi_* \mathscr{L}) - c_1(\operatorname{Sym}^2 \chi_*(\mathscr{L} \otimes \mathscr{P}))$$

For a vector bundle $\mathcal{G}$ we have the elementary fact

$$c_1(\operatorname{Sym}^2 \mathcal{G}) = (\operatorname{rk}(\mathcal{G}) + 1)c_1(\mathcal{G})$$

Furthermore, for every $[C, \eta] \in \overline{\mathcal{R}}_g^0$ and every $L \in W_{16}^4(C)$ we have the vanishing $H^1(C, L^{\otimes 2}) = 0$, so $R^1\chi_*(\mathscr{L}^{\otimes 2}) = 0$. We can then apply Grothendieck–Riemann–Roch and express everything in terms of the classes $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$. For instance we have

$$\begin{aligned} c_1(\chi_*(\mathscr{L}^{\otimes 2})) &= \Bigg[\chi_*\left(1 + c_1(\mathscr{L}^{\otimes 2}) + \frac{c_1^2(\mathscr{L}^{\otimes 2})}{2}\right) \\ &\qquad \cdot \left(1 - \frac{c_1(\omega_X)}{2} + \frac{c_1^2(\omega_X) + c_2(\Omega_X)}{12}\right)\Bigg]_1 \\ &= \lambda + 2\mathfrak{a} - \mathfrak{b} \end{aligned}$$

where $[-]_1$ denotes the degree 1 part of an expression. We have used Mumford's formula to calculate $\chi_*(c_1^2(\omega_X) + c_2(\Omega_X)) = 12\lambda$. Similarly, also using Lemma 2.4, we find

$$c_1(\chi_*(\mathscr{L} \otimes \mathscr{P})) = \lambda + \frac{\mathfrak{a}}{2} - \frac{\mathfrak{b}}{2} - \frac{\delta_0^{\mathrm{ram}}}{4} + \mathfrak{d}$$

Using the results of [Far09], in particular Lemmata 2.6 and 2.13 as well as Proposition 2.12, we can calculate the pushforwards of $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ by σ:

$$\begin{aligned} \sigma_*(\mathfrak{a}) &= -146784\lambda + 20856(\delta_0' + \delta_0'') + 41712\delta_0^{\mathrm{ram}} \\ \sigma_*(\mathfrak{b}) &= 4224\lambda + 264(\delta_0' + \delta_0'') + 528\delta_0^{\mathrm{ram}} \\ \sigma_*(\mathfrak{c}) &= -48279\lambda + 6930(\delta_0' + \delta_0'') + 13860\delta_0^{\mathrm{ram}} \end{aligned}$$

and of course $\sigma_*(\lambda) = N\lambda$, $\sigma_*(\delta_0^{\mathrm{ram}}) = N\delta_0^{\mathrm{ram}}$ where $N = 6006$ is the degree of σ. Putting everything together, we obtain the result. The difference between $[\overline{\mathfrak{D}}_{15}]^{\mathrm{virt}}$ and $[\overline{\mathfrak{D}}_{15}]$ arises from the boundary components where ϕ is degenerate. ∎

Theorem 2.13. *$\overline{\mathcal{R}}_{15,2}$ is of general type.*

Proof. The contribution of $\sigma_*(\mathfrak{d})$ to $[\overline{\mathfrak{D}}_{15}]$ only improves the ratio between the coefficients of λ and the boundary components. The same goes for the boundary components where ϕ is degenerate. Hence we may as well work with the class $[\overline{\mathfrak{D}}_{15}]^{\mathrm{virt}} + 3\sigma_*(\mathfrak{d})$. Then we take an appropriate linear combination of $\overline{\mathfrak{D}}_{15}$ and the divisor $\overline{\mathfrak{D}}_{15:2}$ from [FL10] having class

$$\begin{aligned} [\overline{\mathfrak{D}}_{15:2}] &= 5808\lambda - 924(\delta_0' + \delta_0'') - 990\delta_0^{\mathrm{ram}} \\ &= 924\left(\frac{44}{7}\lambda - (\delta_0' + \delta_0'') - \frac{15}{14}\delta_0^{\mathrm{ram}}\right) \end{aligned}$$

For instance we have

$$\beta\overline{\mathfrak{D}}_{15:2} + \gamma\overline{\mathfrak{D}}_{15} = \epsilon\lambda - 2(\delta_0' + \delta_0'') - 3\delta_0^{\mathrm{ram}}$$

where

$$\beta = \frac{667}{680394}, \quad \gamma = \frac{4}{113399}, \quad \epsilon = \frac{10288}{793}$$

Here $\epsilon < 13$, hence the canonical class is big. ∎

Remark 2.14. The map

$$\operatorname{Sym}^2 H^0(C, L \otimes \eta) \to H^0(C, L^{\otimes 2}) / \operatorname{Sym}^2 H^0(C, L)$$

is identically zero along the boundary component Δ_0''. Hence the morphism ϕ is degenerate with order 3 along Δ_0''. It follows that we can subtract $3\delta_0''$ from $Z_1(\phi)$ and still obtain an effective class.

CHAPTER 3

Twists of Mukai bundles and the Kodaira dimension of $\mathcal{R}_{8,3}$

S. Mukai's celebrated results from [Muk93] show that a general curve of genus 8 is a linear section of the 8-dimensional Grassmannian $G(2,6)$ in $\mathbb{P}^{14}$. In a similar fashion, the general genus 6 curve is the complete intersection of a 4-dimensional quadric and the 6-dimensional Grassmannian $G(2,5)$ in $\mathbb{P}^9$. This was discussed in more detail in section 1.11.

In both cases, the maps from the curve C to the Grassmannian are induced by the global sections of an (up to isomorphism) uniquely determined stable rank 2 bundle E_C with canonical determinant, which we call the *Mukai bundle* of C (see sections 1.10 and 1.11). We have $h^0(C, E_C) = 5$ in genus 6 and $h^0(C, E_C) = 6$ in genus 8. Since the bundle E_C captures the geometry of C, it is a natural problem to study loci of curves where E_C shows non-generic behaviour. In particular, we are interested in divisorial conditions involving E_C on moduli spaces of curves.

We let ℓ be a prime number, $g = 6$ or $g = 8$, C a general curve of genus g and $\eta \in \mathrm{Pic}^0(C)[\ell]$ a line bundle of order ℓ. Then we can ask about the space of sections $H^0(C, E_C \otimes \eta)$. Since the slope of E_C is $g-1$, we expect $H^0(C, E_C \otimes \eta) = 0$ and the locus

$$\left\{[C] \in \mathcal{M}_g \;\middle|\; H^0(E_C \otimes \eta) \neq 0 \text{ for some } \eta \in \mathrm{Pic}^0(C)[\ell] \setminus \{\mathcal{O}_C\}\right\}$$

to be a divisor. In fact it is more natural to study the question on the modular variety $\mathcal{R}_{g,\ell}$. On this space we define the locus

$$\mathcal{B}_{g,\ell} = \left\{[C,\eta] \in \mathcal{R}_{g,\ell} \;\middle|\; H^0(C, E_C \otimes \eta) \neq 0\right\}$$

which is of codimension at most one in $\mathcal{R}_{g,\ell}$ and expected to be a divisor. This we prove:

Theorem 3.1. *In both $g = 6$ and $g = 8$ and for every prime ℓ the locus $\mathcal{B}_{g,\ell}$ is a divisor in $\mathcal{R}_{g,\ell}$.*

Our main interest lies in furthering the understanding of the birational geometry of $\mathcal{R}_{g,\ell}$. In section 1.8 we saw that $\mathcal{R}_{g,3}$ is known to be of general type for $g \geqslant 12$ (as proven in [CEFS13]). We can use the divisor $\mathcal{B}_{8,3}$ just constructed to prove the following:

Theorem 3.2. *$\overline{\mathcal{R}}_{8,3}$ is of general type.*

Note that we now have a result for genus 8 and level 3 while there is currently nothing known about $\mathcal{R}_{9,3}$ and $\mathcal{R}_{10,3}$. The Kodaira dimension of $\mathcal{R}_{11,3}$ is at least 19 (proved in [CEFS13]) but our theorem actually suggests that all three spaces could be of general type as well.

The method of obtaining general type results is outlined in section 1.6. It revolves around constructing divisors with a divisor class in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{R}}_{g,\ell})$ satisfying certain numerical bounds. Hence the first step is to calculate the divisor class of the closure $\overline{\mathcal{B}}_{g,\ell}$ in a partial compactification of $\mathcal{R}_{g,\ell}$. We calculate the class for both $g = 6$ and 8 and for all ℓ. The appropriate partial compactification $\mathcal{R}'_{g,\ell}$ of $\mathcal{R}_{g,\ell}$ which allows us to do so contains only smooth and irreducible one-nodal curves. Over $\mathcal{R}'_{g,\ell}$ we express the closure of $\mathcal{B}_{g,\ell}$ as the degeneracy locus of a morphism $\phi_{g,\ell}$ between vector bundles of the same rank. Using Porteous' formula and the machinery for calculating Chern classes of vector bundles over $\overline{\mathcal{M}}_g$, developed in [Far09], we then show:

Theorem 3.3. *We have the following expressions for the classes of the degeneracy loci of $\phi_{g,\ell}$:*

a) The virtual class of the closure of $\mathcal{B}_{6,\ell}$ in $\mathcal{R}'_{6,\ell}$ is given by

$$[\overline{\mathcal{B}}_{6,\ell}]^{\mathrm{virt}} = 35\lambda - 5(\delta_0' + \delta_0'') - \frac{5}{\ell}\sum_{a=1}^{\lfloor \ell/2 \rfloor}(\ell^2 - a\ell + a^2)\delta_0^{(a)}$$

b) The virtual class of the closure of $\mathcal{B}_{8,\ell}$ in $\mathcal{R}'_{8,\ell}$ is given by

$$[\overline{\mathcal{B}}_{8,\ell}]^{\mathrm{virt}} = 196\lambda - 28(\delta_0' + \delta_0'') - \frac{14}{\ell}\sum_{a=1}^{\lfloor \ell/2 \rfloor}(2\ell^2 - a\ell + a^2)\delta_0^{(a)}$$

In particular, the classes $[\overline{\mathcal{B}}_{g,\ell}]^{\mathrm{virt}} - [\overline{\mathcal{B}}_{g,\ell}]$ are effective and entirely supported on the boundary of $\mathcal{R}'_{g,\ell}$.

By describing the degeneracy of the morphism used in Porteous' formula along the boundary we can improve these divisor classes still further. Similarly, we also improve a divisor class found in [CEFS13]. Combining these results we can prove our Main Theorem 3.2.

3.1 Recap on Mukai bundles

Let $g = 6$ or $g = 8$. We will always denote the Mukai bundle associated to a curve C by E_C. Recall from section 1.11 that by results of [Muk93] it is possible to give explicit Brill–Noether type conditions for a curve to arise as a section of a Grassmannian. We get the vector bundles E_C in question by restricting the tautological bundle of the Grassmannian to C. Importantly for us, it turns out that the existence of a vector bundle with the right numerics is guaranteed by slightly weaker assumptions:

Theorem 3.4 ([Muk93, §5])**.** *Let C be a curve of genus 6 which is neither trigonal nor a plane quintic. When F runs over all stable rank 2 bundles with canonical determinant on C, the maximum of $h^0(C, F)$ is equal to 5. Moreover, such vector bundles E_C on C with $h^0(C, E_C) = 5$ are unique up to isomorphism and generated by global sections.*

Theorem 3.5 ([Muk93, §3])**.** *Let C be a curve of genus 8 without a $\mathfrak{g}^1_4$. When F runs over all semistable rank 2 bundles with canonical determinant on C, the maximum of $h^0(C, F)$ is equal to 6. Moreover, such vector bundles E_C on C with $h^0(C, E_C) = 6$ are unique up to isomorphism and generated by global sections.*

Note the difference to Theorem 1.54 where we restricted to curves having no $\mathfrak{g}^2_7$, but got stability of E_C in return. We also remark that every tetragonal genus 8 curve has a $\mathfrak{g}^2_7$ ([Muk93, Lemma 3.8]).

We denote the locus of curves satisfying the assumptions of Theorem 3.4 or 3.5 by $\mathcal{M}^{\mu}_g$ and we set $\mathcal{R}^{\mu}_{g,\ell} = \mathcal{M}^{\mu}_g \times_{\mathcal{M}_g} \mathcal{R}_{g,\ell}$. The codimension of the complement of this locus is two: In genus 6 the trigonal locus has codimension 2 and the locus of plane quintics has codimension 3. In genus 8, the tetragonal locus is also of codimension 2.

3.2 Constructing the divisors

In both genera, the slope of E_C is

$$\mu(E_C) = \frac{\deg \det E_C}{\mathrm{rk}\, E_C} = \frac{2g-2}{2} = g - 1$$

hence $\chi(E_C) = 0$. We can therefore consider the virtual theta divisor of E_C

$$\Theta_{E_C} = \left\{ \xi \in \mathrm{Pic}^0(C) \;\middle|\; H^0(C, E_C \otimes \xi) \neq 0 \right\}$$

in $\mathrm{Pic}^0(C)$. Since E_C is a semistable rank 2 vector bundle, Θ_{E_C} is indeed of codimension one (see [Ray82, Proposition 1.6.2]). By intersecting $\mathcal{R}_{g,\ell}$ and the locus $\{[C, \xi] \mid \xi \in \Theta_{E_C}\}$ in the universal Jacobian, we expect that

$$\mathcal{B}_{g,\ell} = \left\{ [C, \eta] \in \mathcal{R}_{g,\ell} \;\middle|\; H^0(C, E_C \otimes \eta) \neq 0 \right\}$$

is a divisor as well. Since $\mathcal{R}_{g,\ell}$ is irreducible, by semicontinuity it is enough to exhibit a single pair $[C, \eta]$ such that $H^0(C, E_C \otimes \eta) = 0$. We are going to do this separately for both genera. It will be necessary first to give other characterizations of the pairs $[C, \eta] \in \mathcal{B}_{g,\ell}$. We will also need the following theorem on torsion bundles on hyperelliptic curves:

Theorem 3.6 ([CEFS13, Theorem 2.3]). *Let $[C, p] \in \mathcal{M}_{g,1}$ be a general hyperelliptic curve of genus $g \geqslant 2$ together with a Weierstrass point. Then there exists a torsion point $\eta \in \mathrm{Pic}^0(C)[\ell] \setminus \{\mathcal{O}_C\}$ such that*

$$H^0\big(C, \eta \otimes \mathcal{O}_C((g-1)p)\big) = 0$$

Remark 3.7. In the situation of the previous theorem, $\omega_C = \mathcal{O}_C((2g-2)p)$ and $\chi(\eta \otimes \mathcal{O}_C((g-1)p)) = 0$. Using Serre duality, we also get

$$H^0\big(C, \eta^{-1} \otimes \mathcal{O}_C((g-1)p)\big) = 0$$

3.2.1 Reinterpreting the divisor

In the following discussion we will only consider $[C, \eta] \in \mathcal{R}^{\mu}_{g,\ell}$. As discussed in Lemma 1.55, the bundle E_C is an extension

$$0 \to A \to E_C \to \omega_C \otimes A^{-1} \to 0$$

where $A \in W^1_4(C)$ if $g = 6$ and $A \in W^1_5(C)$ if $g = 8$. After tensoring with η, the associated long exact sequence in cohomology starts with

$$0 \to H^0(C, A \otimes \eta) \to H^0(C, E_C \otimes \eta) \to H^0(C, \omega_C \otimes A^{-1} \otimes \eta) \xrightarrow{\delta_{E_C \otimes \eta}} H^1(C, A \otimes \eta)$$

We immediately get:

Lemma 3.8. $[C, \eta] \in \mathcal{B}_{g,\ell}$ *if and only if there exists an* A *such that* $H^0(C, A \otimes \eta) \neq 0$ *or the boundary map*

$$\delta_{E_C \otimes \eta} \colon H^0(C, \omega_C \otimes A^{-1} \otimes \eta) \to H^1(C, A \otimes \eta) \tag{3.1}$$

is not an isomorphism.

Since $H^0(C, A \otimes \eta) \neq 0$ happens only on curves in a subvariety of codimension at least 2, in what follows we will ignore the locus of such curves.

In genus 6, we can give another interpretation of $\mathcal{B}_{6,\ell}$. Let $A \in W^1_4(C)$ and $L = \omega_C \otimes A^{-1}$. By Riemann–Roch, we have $h^0(C, L \otimes \eta) = 1$ and also $h^1(C, A \otimes \eta) = 1$. So for (3.1) to be an isomorphism it is enough for it to be nonzero.

Lemma 3.9. *In the case $g = 6$, the boundary map $\delta_{E_C}\colon H^0(C, L\otimes\eta) \to H^1(C, A\otimes\eta)$ is nonzero if and only if the multiplication map followed by projection*

$$H^0(C, L \otimes \eta) \otimes H^0(C, L \otimes \eta^{-1}) \xrightarrow{m_\eta} H^0(C, L^{\otimes 2}) \xrightarrow{p} H^0(C, L^{\otimes 2})/\operatorname{Sym}^2 H^0(C, L) \tag{3.2}$$

is an isomorphism.

Proof. Since C is not a plane quintic, L is base point free, so it induces a morphism to $\mathbb{P}^2$. The image is birational to C if and only if C is not trigonal and not bielliptic, so for a general genus 6 curve L induces a birational map to a 4-nodal plane sextic. This implies that the multiplication map $\operatorname{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ is injective. So both domain and codomain of the map $p \circ m_\eta$ are 1-dimensional. For bielliptic C the same conclusion holds by $H^0(C, L) \cong H^0(E, \mathcal{O}_E(1))$. Hence (3.2) is an isomorphism if and only if it is nonzero.

Extensions of L by A and of $L \otimes \eta$ by $A \otimes \eta$ are both parametrized by

$$\operatorname{Ext}^1(L, A) \cong \operatorname{Ext}^1(L \otimes \eta, A \otimes \eta) \cong H^1(C, A \otimes L^{-1}) \cong H^0(C, L^{\otimes 2})^\vee$$

while the boundary morphism $\delta_{E_C \otimes \eta}$ lives in

$$\begin{aligned}\operatorname{Hom}(H^0(C, L \otimes \eta), H^1(C, A \otimes \eta)) &\cong H^0(C, L \otimes \eta)^\vee \otimes H^1(C, A \otimes \eta)\\ &\cong H^0(C, L \otimes \eta)^\vee \otimes H^0(C, L \otimes \eta^{-1})^\vee\end{aligned}$$

and we have a map

$$\alpha\colon H^0(C, L^{\otimes 2})^\vee \to H^0(C, L \otimes \eta)^\vee \otimes H^0(C, L \otimes \eta^{-1})^\vee \tag{3.3}$$

sending an extension $E \otimes \eta$ to the boundary homomorphism $\delta_{E\otimes\eta}$. Note that α is the dual of the multiplication map m_η. We denote by $[\alpha]$ the composition of α with the dual of the projection p.

The space $H^0(C, L^{\otimes 2})^\vee/\operatorname{Sym}^2 H^0(C, L)^\vee$ is generated by the class $[\phi_{E_C}]$ of the map corresponding to the Mukai bundle E_C (see the discussion after Lemma 1.55). Now (3.2) is the zero map if and only if the dual map $[\alpha]$ is the zero map if and only if $[\phi_{E_C}]$ is mapped to 0 by $[\alpha]$, i.e., if $[\phi_{E_C}] \circ (p \circ m_\eta) = 0$. But this is exactly the boundary map $\delta_{E_C\otimes\eta}$ given by the image of the extension $E_C \otimes \eta$ under (3.3). ∎

Remark 3.10. For the case $\ell = 2$ further descriptions of the divisor exist. A general curve $[C, \eta] \in \mathcal{B}_{6,2}$ equivalently satisfies the following conditions:

a) C has a 4-nodal plane sextic model with a totally tangent conic, i.e., there exists an $L \in W^2_6(C)$ inducing a birational map to $\Gamma \subseteq \mathbb{P}^2$, and a conic $Q \subseteq \mathbb{P}^2$ with $Q \cap \Gamma = 2D$ for some $D \in C^{(6)}$. This identification follows from Lemma 3.9.

b) The Prym map $\mathcal{R}_{6,2} \to \mathcal{A}_5$ is ramified at $[C, \eta]$ (see [FGSV14, Theorem 8.1]).

c) $[C, \eta]$ is in the Prym–Brill–Noether divisor in $\mathcal{R}_{6,2}$, i.e.,

$$\emptyset \neq V_3(C, \eta) = \left\{ L \in \mathrm{Nm}_f^{-1}(K_C) \,\middle|\, \begin{array}{l} h^0(\widetilde{C}, L) \geqslant r+1, \\ h^0(\widetilde{C}, L) \equiv r+1 \pmod 2 \end{array} \right\}$$

where $f\colon \widetilde{C} \to C$ is the étale double cover associated to η ([FGSV14, Theorem 0.4]).

d) $[C, \eta]$ is a section of a Nikulin surface (see [FV12, Theorem 0.5]).

3.2.2 Proof of transversality for genus 6

Recall that our aim is to show $H^0(C, E_C \otimes \eta) = 0$ for the general pair $[C, \eta] \in \mathcal{R}^{\mu}_{6,\ell}$. To prepare the proof, we first show the following lemma.

Lemma 3.11. *Let $C \in \mathcal{M}_6$ be a general plane quintic. Then there exists a line bundle $\eta \in \mathrm{Pic}^0(C)[\ell]$ such that*

$$H^0(C, \mathcal{O}_C(1) \otimes \eta) = H^0(C, \mathcal{O}_C(1) \otimes \eta^{-1}) = 0$$

Proof. Denote by Q_6 the locus of plane quintics in $\mathcal{M}_6$. In [Gri85] it is proved that the closure $\overline{Q_6}$ of Q_6 in $\mathcal{M}_6$ is

$$\overline{Q_6} = Q_6 \cup \mathcal{H}_6$$

where $\mathcal{H}_6$ is the hyperelliptic locus. It is also shown that $\mathfrak{G}^2_5$, the universal $\mathfrak{g}^2_5$ over $\mathcal{M}_6$, consists of two components W_1 and W_2 where W_1 parametrizes plane quintics together with their unique $\mathfrak{g}^2_5$ while W_2 parametrizes hyperelliptic curves Y together with a line bundle $\mathcal{O}_Y(4p + q)$. Here $p, q \in Y$ and p is a Weierstrass point. The intersection of the two components is given by

$$W_1 \cap W_2 = \{\, [Y, \mathcal{O}_Y(5p)] \mid Y \in \mathcal{H}_6, p \in Y \text{ a Weierstrass point}\}$$

We can therefore take the étale cover $\mathfrak{G}^{2,(\ell)}_5 = \mathfrak{G}^2_5 \times_{\mathcal{M}_6} \mathcal{R}_{6,\ell}$ of $\mathfrak{G}^2_5$ (which might be highly reducible) and consider a hyperelliptic curve $[Y, \mu, \mathcal{O}_Y(5p)]$ with $p \in Y$ a Weierstrass point and μ an ℓ-torsion line bundle such that we have the vanishing $H^0(Y, \mu \otimes \mathcal{O}_Y(5p)) = 0$. Such a triple exists by Theorem 3.6. Then we specialize a plane quintic $[C, \eta, \mathcal{O}_C(1)]$ in the correct component to $[Y, \mu, \mathcal{O}_Y(5p)]$. The result follows by semicontinuity. ∎

Remark 3.12. For $\ell = 2$ the argument can be shortened a lot. Let $\vartheta = \mathcal{O}_C(1)$, which is a theta characteristic, and take another, noneffective theta characteristic ζ. Then set $\eta = \zeta \otimes \vartheta^{-1} \in \mathrm{Pic}^0(C)[2]$.

We now exhibit a particular curve $[C,\eta] \in \mathcal{R}_{6,\ell}$ such that the map (3.2) of Lemma 3.9 is an isomorphism for all $L \in W^2_6(C)$. By the irreducibility of $\mathcal{R}_{6,\ell}$ we obtain that the map is an isomorphism for all L on pairs $[C,\eta]$ in an open subset of $\mathcal{R}_{6,\ell}$.

To construct our example, we specialize to a plane quintic C and choose any $L \in W^2_6(C)$. Let $\vartheta = \mathcal{O}_C(1)$ be the unique $\mathfrak{g}^2_5$ on C and recall that it is an odd theta characteristic. Now L can be written as

$$L = \vartheta \otimes \mathcal{O}_C(x)$$

for some point $x \in C$. In particular, x is a base point of L (and in fact the only one). Using Lemma 3.11, choose an ℓ-torsion bundle η on C such that $h^0(C, \vartheta \otimes \eta) = 0$. Then, by Riemann–Roch and Serre duality,

$$h^0(C, \vartheta \otimes \eta^{-1}) = h^1(C, \vartheta \otimes \eta^{-1}) = h^0(C, \omega_C \otimes \vartheta^{-1} \otimes \eta) = h^0(C, \vartheta \otimes \eta) = 0$$

as well. This implies

$$h^0(C, L \otimes \eta) = h^0(C, L \otimes \eta^{-1}) = 1$$

and x is neither a base point of $L \otimes \eta$ nor of $L \otimes \eta^{-1}$. Let $H^0(C, L \otimes \eta) = \langle \sigma \rangle$ and $H^0(C, L \otimes \eta^{-1}) = \langle \tau \rangle$ and consider the map

$$\langle \sigma \rangle \otimes \langle \tau \rangle \to H^0(C, L^{\otimes 2}) / \operatorname{Sym}^2 H^0(C, L)$$

Observe that the multiplication map $\operatorname{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ is injective, since

$$\operatorname{Sym}^2 H^0(C, \mathcal{O}_C(1)) \to H^0(C, \mathcal{O}_C(2))$$

is an isomorphism ($C \subseteq \mathbb{P}^2$ is not contained in a quadric). The base locus of the image of $\operatorname{Sym}^2 H^0(C, L)$ in $H^0(C, L^{\otimes 2}) = H^0(C, \omega_C(2x))$ contains $2x$. But $\sigma \otimes \tau$, considered as a section in $H^0(C, L^{\otimes 2})$, does not vanish at x. Therefore it cannot be contained in the image of $\operatorname{Sym}^2(C, L)$, whence

$$H^0(C, L^{\otimes 2}) \cong \langle \sigma \otimes \tau \rangle \oplus \operatorname{Sym}^2 H^0(C, L)$$

and we are done.

3.2.3 Proof of transversality for genus 8

Again we want to show $H^0(C, E_C \otimes \eta) = 0$ for a general curve $[C, \eta] \in \mathcal{R}^{\mu}_{8,\ell}$. We first specialize to a general curve C having an $L \in W^2_7(C)$ but no $\mathfrak{g}^1_4$. Such curves are constructed, for instance, in [IM03].

By Theorem 3.5 there is a unique semistable rank 2 vector bundle E_C on C with canonical determinant and $h^0(C, E_C) = 6$. On the other hand, $L \oplus (\omega_C \otimes L^{-1})$ is such a vector bundle. We conclude $E_C \cong L \oplus (\omega_C \otimes L^{-1})$. We also obtain the following lemma as a corollary of the uniqueness of E_C.

Lemma 3.13. *If $W^1_4(C) = \emptyset$ and $W^2_7(C) \neq \emptyset$, then $W^2_7(C) = \{L, \omega_C \otimes L^{-1}\}$. If L is autoresidual, there is only one (fat) point.*

For any $\eta \in \mathrm{Pic}^0(C)[\ell]$ it now follows that

$$H^0(C, E_C \otimes \eta) = H^0(C, L \otimes \eta) \oplus H^0(C, \omega_C \otimes L^{-1} \otimes \eta)$$

hence we have to show $H^0(C, L \otimes \eta) = H^0(C, \omega_C \otimes L^{-1} \otimes \eta) = 0$ for some η. In order to do this, we want to specialize $[C, L]$ further to a hyperelliptic curve with a $\mathfrak{g}^2_7$. For this step we need the following result:

Lemma 3.14. *The universal linear series $\mathfrak{G}^2_7$ over the locus $\mathcal{M}^2_{8,7}$ of curves having a $\mathfrak{g}^2_7$ is irreducible.*

Proof. Every component of $\mathfrak{G}^2_7$ has dimension at least $3g - 3 + \rho = 20$ where $\rho = \rho(8, 2, 7)$ is the Brill–Noether number. We also know that since $\rho = -1$, the locus $\mathcal{M}^2_{8,7}$ is an irreducible divisor (see Theorem 1.31). By Theorem 1.43 there is a unique "main component" of $\mathfrak{G}^2_7$ that maps dominantly onto $\mathcal{M}^2_{8,7}$. We can explicitly analyze the candidate components of $\mathfrak{G}^2_7$ that would map to a proper subvariety of $\mathcal{M}^2_{8,7}$. These proper subvarieties consist of curves C where

a) C is hyperelliptic, i.e., has a $\mathfrak{g}^1_2$

b) C is trigonal, i.e., has a $\mathfrak{g}^1_3$

c) C has a $\mathfrak{g}^2_6$

d) C has a $\mathfrak{g}^1_4$

If C is hyperelliptic, we have a 3-dimensional family of linear systems of the form $(\mathfrak{g}^1_2)^{\otimes 2} \otimes \mathcal{O}_C(q_1 + q_2 + q_3)$ and a 3-dimensional family of 2-dimensional linear subseries for each line bundle $(\mathfrak{g}^1_2)^{\otimes 3} \otimes \mathcal{O}_C(q)$ of which we have a one-dimensional family. In total, the dimension of the space of these linear series does not exceed $\dim(\mathcal{H}_8) + 4 = 19$, so it has to be contained in another irreducible component.

If C is not hyperelliptic, then by Mumford's theorem (see [Kee90, Proposition 0.2]), we have $\dim W^2_7(C) \leqslant 2$ with equality if and only if C is trigonal or bielliptic. Since the locus of trigonal curves has dimension 17, the space of linear series over it is at most of dimension 19. The locus of bielliptic curves is of dimension $2g - 2 = 14$ and so the linear series form a family of dimension 16 only.

Now if C is not bielliptic but still has a $\mathfrak{g}^2_6$, we have a one-dimensional family of $\mathfrak{g}^2_7$. But since the locus of curves of with a $\mathfrak{g}^2_6$ is only 17-dimensional (the general such curve can be realized as a 2-nodal degree 6 plane curve), the family can be at most 18-dimensional.

Finally, if C is a general tetragonal curve, then by [CM99, Theorem 3], the dimension of $W^2_7(C)$ is zero. It follows that the family of linear series over the tetragonal locus is exactly 19-dimensional.

Combining everything, only the whole locus $\mathcal{M}^2_{8,7}$ gives rise to an at least 20-dimensional component, which therefore has to be unique. ■

Using the irreducibility of $\mathfrak{G}^2_7$ we may now specialize C further to a general hyperelliptic curve and L to

$$L = (\mathfrak{g}^1_2)^{\otimes 2} \otimes \mathcal{O}_C(q_1 + q_2 + q_3) \cong \mathcal{O}_C(4p + q_1 + q_2 + q_3)$$

where $q_i \in C$ are general points and p is a Weierstrass point. By Theorem 3.6 this $p \in C$ can be chosen such that there exists an $\eta \in \mathrm{Pic}^0(C)[\ell]$ with $H^0(C, \eta \otimes \mathcal{O}_C(7p)) = 0$. It is now clear that $H^0(C, L \otimes \eta) = 0$. The same statement holds for the dual bundle η^{-1} by Remark 3.7 and hence, using $\chi(L) = 0$,

$$h^0(C, \omega_C \otimes L^{-1} \otimes \eta) = h^1(C, L \otimes \eta^{-1}) = h^0(C, L \otimes \eta^{-1}) = 0$$

Since the general curve $C \in \mathcal{M}^2_{8,7}$ has $W^2_7(C)$ finite, we can apply semicontinuity to finish the proof.

3.3 Divisor classes

3.3.1 Strategy

An effective method to calculate divisor classes is to give a determinantal description of the divisors, i.e., express them as the locus where a certain morphism between vector bundles drops rank. If the divisor involves global sections of line bundles on curves, the vector bundles are usually constructed over some space $\mathfrak{G}^r_d$ of linear series over the moduli space of curves.

To calculate the classes of $\mathcal{B}_{g,\ell}$ or some compactification of it, a direct approach would be to try to use Lemma 3.8 and globalize the map

$$\delta_{E_C \otimes \eta} \colon H^0(C, \omega_C \otimes A^{-1} \otimes \eta) \to H^1(C, A \otimes \eta)$$

to a morphism of vector bundles over $\mathfrak{G}^{r,(\ell)}_d = \mathfrak{G}^r_d \times_{\mathcal{M}_g} \mathcal{R}_{g,\ell}$. A naive candidate is to pass to the moduli stacks and to try to create a global extension

$$0 \to \mathcal{A} \to \mathcal{E} \to \omega_\chi \otimes \mathcal{A}^{-1} \to 0,$$

on the universal curve $\chi\colon \mathfrak{C}^{r,(\ell)}_d \to \mathfrak{G}^{r,(\ell)}_d$, tensor it by the universal ℓ-torsion bundle $\mathscr{P}$ and use the map induced by the long exact sequence of the push-forward σ_* where $\sigma\colon \mathfrak{G}^{r,(\ell)}_d \to \mathcal{R}_{g,\ell}$.

However, this naive approach must fail. The bundle E_C, as an extension of $\omega_C \otimes A^{-1}$ by A, is only defined up to isomorphism on each curve and the choice can not be made globally on the whole moduli space. It is possible though to give a choice-free description of the condition that the boundary morphism induced by $E_C \otimes \eta$ is not an isomorphism.

To this end, let $L = \omega_C \otimes A^{-1}$ and observe that the codomain of $\delta_{E_C \otimes \eta}$ is

$$H^1(C, A \otimes \eta) \cong H^0(C, L \otimes \eta^{-1})^\vee$$

by Serre duality. Now the map

$$H^0(C, L \otimes \eta) \otimes \left(\frac{H^0(C, L^{\otimes 2})}{\operatorname{Sym}^2 H^0(C, L)} \right)^\vee \to H^0(C, L \otimes \eta^{-1})^\vee \tag{3.4}$$

can be defined canonically by setting

$$s \otimes f \mapsto [t \mapsto f(s \cdot t)]$$

The quotient that appears can be seen as encoding the $\mathbb{C}^*$ of possible choices for E_C in $\operatorname{Ext}^1(L, A)$. It is clear that the map (3.4) is an isomorphism if and only if $\delta_{E_C \otimes \eta}$ is. Since there are no choices involved in defining the map, we can readily globalize it.

3.3.2 Definition of the degeneracy locus

We use a setup similar to [FL10] and [CEFS13]. Let $\pi\colon \overline{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{M}}_g$ be the forgetful map. Using Lemma 1.26 we can restrict our attention to the boundary divisors Δ_0', Δ_0'' and $\Delta_0^{(a)}$ of $\overline{\mathcal{R}}_{g,\ell}$ when calculating divisor classes.

The first step is to construct an appropriate partial compactification of $\mathcal{R}_{g,\ell}$ where the class calculations can be carried out. Let $\mathcal{R}'_{6,\ell} = \mathcal{R}^0_{6,\ell} \cup \pi^*(\Delta_0^0)$, where $\mathcal{R}^0_{6,\ell}$ is the locus of smooth curves $[C, \eta]$ such that $\dim W^2_6(C) = 0$ and $H^1(C, L \otimes \eta) = 0$ for all $L \in W^2_6(C)$, and Δ_0^0 is the locus of irreducible one-nodal curves $[C_{pq}] \in \Delta_0$ where $[C, p, q] \in \mathcal{M}_{5,2}$ is Petri general.

Similarly, let $\mathcal{R}'_{8,\ell} = \mathcal{R}^0_{8,\ell} \cup \pi^*(\Delta_0^0)$ be the locus of smooth curves $[C, \eta] \in \mathcal{R}_{8,\ell}$ such that $\dim W^3_9(C) = 0$, and $H^1(C, L \otimes \eta) = 0$ for all $L \in W^3_9(C)$, while Δ_0^0 is the locus of curves $[C_{pq}]$ with $[C, p, q] \in \mathcal{M}_{7,2}$ Petri general. Observe that in both cases the complement of $\mathcal{R}'_{g,\ell}$ in $\mathcal{R}_{g,\ell} \cup \pi^*(\Delta_0)$ has codimension 2, so divisor class calculations will not be affected.

We are now in a position to provide a determinantal description of the divisor $\mathcal{B}_{g,\ell}$. To this end, we will construct a morphism of vector bundles of the same rank over $\mathcal{R}'_{g,\ell}$ such that on fibers it corresponds exactly to the map in (3.4). Then $\overline{\mathcal{B}}_{g,\ell}$ will be contained in the first degeneracy locus of this morphism and its class can be calculated using Porteous' formula.

The set-up is almost the same for both genera. In genus 6 we let $r = 2$, $d = 6$ and in genus 8 we let $r = 3$, $d = 9$. Now let $\mathfrak{G}^{r,(\ell)}_d$ be the moduli stack of triples $[C, \eta, L]$ over $\mathcal{R}'_{g,\ell}$ where $L \in W^r_d(C)$ and let $\sigma\colon \mathfrak{G}^{r,(\ell)}_d \to \mathcal{R}'_{g,\ell}$ be the morphism forgetting the $\mathfrak{g}^r_d$. Denote further by $\chi\colon \mathfrak{C}^{r,(\ell)}_d \to \mathfrak{G}^{r,(\ell)}_d$ the universal curve and let $\mathscr{L}$ be the universal $\mathfrak{g}^r_d$. We also have the universal ℓ-torsion bundle $\mathscr{P}$ over $\mathfrak{C}^{r,(\ell)}_d$. We will slightly abuse notation and denote the pullbacks of λ, δ_0', δ_0'' and $\delta_0^{(a)}$ by σ by the same symbols, respectively.

By Grauert's theorem, $\chi_*(\mathscr{L}^{\otimes i})$ is a vector bundle for $i = 1, 2$. However, we do not know this for $\chi_*(\mathscr{L} \otimes \mathscr{P})$, since the dimension of $H^0(C, L \otimes \eta)$ jumps on fibers over the whole boundary divisor Δ_0'':

Lemma 3.15. *Let* $g = 6$ *and* $[C, \eta] \in \Delta_0''$. *Then for any* $L \in W^2_6(C)$ *we have* $h^0(C, L \otimes \eta) = 2$. *Likewise, for* $g = 8$ *and any* $L \in W^3_9(C)$ *on* $[C, \eta] \in \Delta_0''$ *we have* $h^0(C, L \otimes \eta) = 3$.

Proof. Let $\nu\colon \widetilde{C} \to C$ be the normalization of C and x be the node. Then $\nu^*\eta = \mathcal{O}_{\widetilde{C}}$ and $\nu^*L \in W^r_d(\widetilde{C})$, since $\widetilde{C}$ is Brill–Noether general. From the exact sequence

$$0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_{\widetilde{C}} \xrightarrow{e} \mathbb{C}_x \to 0$$

we get

$$0 \to L \otimes \eta \to \nu_*\nu^*L \xrightarrow{e'} L \otimes \eta|_x \to 0$$

and taking long exact sequence in cohomology we obtain

$$0 \to H^0(C, L \otimes \eta) \to H^0(\widetilde{C}, \nu^*L) \xrightarrow{H^0(e')} \mathbb{C}$$

Now $H^0(e)$ is the zero map, hence $H^0(e')$ must be nonzero and we get

$$h^0(C, L \otimes \eta) = h^0(\widetilde{C}, \nu^*L) - 1 = r + 1 - 1 = r$$ ∎

This shows that $R^1\chi_*(\mathscr{L} \otimes \mathscr{P})$ is supported on Δ_0'' and there it is of rank 1. On the other hand, we do not know whether $\chi_*(\mathscr{L} \otimes \mathscr{P})$ is a vector bundle. However, it is torsion-free since it is a subbundle of $\chi_*(\mathscr{L} \otimes \mathscr{P} \otimes \omega_\chi)$, which is locally free by Grauert's theorem. Hence $\chi_*(\mathscr{L} \otimes \mathscr{P})$ is locally free in codimension 1 and we can throw out the loci of codimension at least 2 in $\mathcal{R}'_{g,\ell}$ where the rank jumps. This will not affect our divisor class calculations. Hence we will assume $\chi_*(\mathscr{L} \otimes \mathscr{P})$ and $\chi_*(\mathscr{L} \otimes \mathscr{P}^{-1})$ are vector bundles. Now let

$$\mathcal{E} = \chi_*(\mathscr{L} \otimes \mathscr{P}) \otimes \left(\chi_*(\mathscr{L}^{\otimes 2}) / \operatorname{Sym}^2 \chi_*(\mathscr{L})\right)^\vee$$

and

$$\mathcal{F} = \left(\chi_*(\mathscr{L} \otimes \mathscr{P}^{-1})\right)^\vee$$

We obtain a morphism

$$\phi_{g,\ell}\colon \mathcal{E} \to \mathcal{F} \tag{3.5}$$

whose first degeneracy locus $Z_1(\phi_{g,\ell})$, pushed forward by σ and restricted to the Mukai locus $\mathcal{R}^\mu_{g,\ell}$, coincides with our divisor $\mathcal{B}_{g,\ell}$.

3.3.3 Calculation of the classes

First we apply Porteous' formula to the morphism (3.5) to obtain

$$[Z_1(\phi_{g,\ell})] = c_1(\mathcal{F} - \mathcal{E}) = c_1(\mathcal{F}) - c_1(\mathcal{E})$$

Using the elementary fact

$$c_1(\operatorname{Sym}^2 \mathcal{G}) = (\operatorname{rk}(\mathcal{G}) + 1)c_1(\mathcal{G})$$

for a vector bundle $\mathcal{G}$ we can write

$$c_1(\mathcal{E}) = c_1(\chi_*(\mathscr{L} \otimes \mathscr{P})) - (r-1)c_1(\chi_*(\mathscr{L}^{\otimes 2})) + (r-1)(r+2)c_1(\chi_*(\mathscr{L}))$$

We use Grothendieck–Riemann–Roch to calculate the Chern classes in these expressions. Let ω_χ be the relative dualizing sheaf of χ and consider the classes

$$\mathfrak{a} = \chi_*(c_1^2(\mathscr{L})), \quad \mathfrak{b} = \chi_*(c_1(\mathscr{L}) \cdot c_1(\omega_\chi)), \quad \mathfrak{c} = c_1(\chi_*(\mathscr{L}))$$

in $A^1(\mathfrak{G}_d^{r,(\ell)})$. Furthermore, let $\mathfrak{d} = c_1\left(R^1\chi_*(\mathscr{L} \otimes \mathscr{P})\right)$. For brevity, set

$$\rho = \sum_{a=1}^{\lfloor \ell/2 \rfloor} \frac{a(\ell - a)}{\ell} \delta_0^{(a)}$$

Applying Grothendieck–Riemann–Roch and using [CEFS13, Proposition 1.6], we get

$$c_1(\chi_*(\mathscr{L} \otimes \mathscr{P}^{\pm 1})) = \lambda + \frac{1}{2}\mathfrak{a} - \frac{1}{2}\mathfrak{b} - \frac{1}{2}\rho + \mathfrak{d}$$
$$c_1(\chi_*(\mathscr{L}^{\otimes 2})) = \lambda + 2\mathfrak{a} - \mathfrak{b}$$

Putting everything together, we obtain

$$[Z_1(\phi_{g,\ell})] = (r-3)\lambda + (2r-3)\mathfrak{a} - (r-2)\mathfrak{b} - (r^2+r-2)\mathfrak{c} - 2\mathfrak{d} + \rho \tag{3.6}$$

Lemma 3.16. *For $g = 6$ we have*

$$\sigma_*(\mathfrak{a}) = -93\,\lambda + \frac{23}{2}\pi^*(\delta_0)$$
$$\sigma_*(\mathfrak{b}) = -\frac{3}{2}\lambda + \frac{3}{4}\pi^*(\delta_0)$$
$$\sigma_*(\mathfrak{c}) = -\frac{133}{4}\lambda + \frac{33}{8}\pi^*(\delta_0)$$

and for $g = 8$ we have

$$\sigma_*(\mathfrak{a}) = -267\,\lambda + \frac{69}{2}\pi^*(\delta_0)$$
$$\sigma_*(\mathfrak{b}) = 3\,\lambda + \frac{3}{2}\pi^*(\delta_0)$$
$$\sigma_*(\mathfrak{c}) = -100\,\lambda + 13\pi^*(\delta_0)$$

Proof. Use the machinery of [Far09], in particular Lemma 2.6, Lemma 2.13 and Proposition 2.12. ∎

Remark 3.17. A different choice of Poincaré bundle $\mathscr{L}$ affects the classes $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$. However, the class of the degeneracy locus of $\phi_{g,\ell}$ is independent of this choice (see the discussion before [Far09, Theorem 2.1]).

Now we only need to pushforward $[Z_1(\phi_{g,\ell})]$ by σ to $\mathcal{R}'_{g,\ell}$, which has the effect of multiplying the coefficients of λ, δ_0'' and $\delta_0^{(a)}$ in (3.6) by the degree of σ. This is 5 in the case of $g = 6$ and 14 in the case of $g = 8$ (the respective number of of $\mathfrak{g}^r_d$ on the general curve). Plug in the expressions of Lemma 3.16 to obtain:

Theorem 3.18. *The class of the degeneracy locus $\sigma_* Z_1(\phi_{6,\ell})$ is*

$$[\overline{\mathcal{B}}_{6,\ell}]^{\mathrm{virt}} = 35\lambda - 5(\delta_0' + 3\delta_0'') - \frac{5}{\ell}\sum_{a=1}^{\lfloor \ell/2 \rfloor}(\ell^2 - a\ell + a^2)\delta_0^{(a)}$$

Theorem 3.19. *The class of the degeneracy locus $\sigma_* Z_1(\phi_{8,\ell})$ is*

$$[\overline{\mathcal{B}}_{8,\ell}]^{\mathrm{virt}} = 196\lambda - 28(\delta_0' + 2\delta_0'') - \frac{14}{\ell}\sum_{a=1}^{\lfloor \ell/2 \rfloor}(2\ell^2 - a\ell + a^2)\delta_0^{(a)}$$

In particular, since $\sigma_* Z_1(\phi_{g,\ell}) \cap \mathcal{R}_{g,\ell} = \mathcal{B}_{g,\ell}$, the class $[\overline{\mathcal{B}}_{g,\ell}]^{\mathrm{virt}} - n[\overline{\mathcal{B}}_{g,\ell}]$ is effective and entirely supported on the boundary of $\mathcal{R}'_{g,\ell}$ for some $n \geqslant 1$.

Remark 3.20. The morphism $\phi_{g,\ell}$ is degenerate over the boundary component Δ_0'', with order 1 for $g = 6$ and order 2 for $g = 8$. We can therefore subtract an additional $5\delta_0''$ and $28\delta_0''$, respectively.

Remark 3.21. The coefficients appearing in the expression of $\overline{\mathcal{B}}_{6,\ell}$ are divisible by 5, which is exactly the degree of the map $\sigma\colon \mathfrak{G}_6^{2,(\ell)} \to \mathcal{R}'_{6,\ell}$. This can be explained by observing that the boundary morphism (3.1) fails to be an isomorphism for some $A \in W^1_4(C)$ if and only if $H^0(C, E_C \otimes \eta) \neq 0$. But since E_C does not depend on the choice of A, the morphism surprisingly fails to be bijective for *all* $A \in W^1_4(C)$.

Similarly, the coefficients for $\overline{\mathcal{B}}_{8,\ell}$ are divisible by $28 = 2 \cdot 14$, where 14 is the degree of σ. Observe that by Serre duality, $\chi(E_C \otimes \eta) = 0$ and the isomorphism $E_C^\vee \otimes \omega_C \cong E_C$ we have $H^0(C, E_C \otimes \eta) = 0$ if and only if $H^0(C, E_C \otimes \eta^{-1}) = 0$. This explains the additional factor of two.

3.4 Application to the birational geometry of modular varieties

3.4.1 An improvement of existing divisor classes

Recall the following result:

Theorem 3.22 ([CEFS13, Theorem 0.7])**.** *Set $g = 2i + 2 \geqslant 4$ and $\ell \geqslant 3$ such that $i \equiv 1 \mod 2$ or $\binom{2i-1}{i} \equiv 0 \mod 2$. The virtual class of the closure in $\mathcal{R}'_{g,\ell}$ of the locus $\mathcal{D}_{g,\ell}$ of level ℓ curves $[C, \eta] \in \mathcal{R}_{g,\ell}$ such that $K_{i,1}(C; \eta^{\otimes(\ell-2)}, K_C \otimes \eta) \neq 0$ is equal to*

$$[\overline{\mathcal{D}}_{g,\ell}]^{\mathrm{virt}} = \frac{1}{i-1}\binom{2i-2}{i}\Bigg((6i+1)\lambda - i(\delta_0' + \delta_0'') \\ - \frac{1}{\ell}\sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor}(i\ell^2 + 5a^2 i - 5ai\ell - 2a^2 + 2a\ell)\delta_0^{(a)}\Bigg)$$

We quickly sketch how this result was obtained. By standard arguments, e.g., [AN10], we have an identification

$$K_{i,1}(C; \eta^{\otimes(\ell-2)}, K_C \otimes \eta) = H^0(C, \wedge^i M_{K_C \otimes \eta} \otimes K_C \otimes \eta^{-1})$$

This in turn can be identified with the kernel of the map

$$\wedge^i H^0(C, K_C \otimes \eta) \otimes H^0(C, K_C \otimes \eta^{-1}) \to H^0(C, \wedge^{i-1} M_{K_C \otimes \eta} \otimes K_C^{\otimes 2})$$

Note that the domain and the target are vector spaces of the same dimension. This map is then globalized to a map χ between vector bundles of the same rank over $\mathcal{R}'_{g,\ell}$ and its first degeneracy locus can be calculated using Porteous' formula to obtain the class of Theorem 3.22.

We will now show that the map χ is degenerate along all the boundary divisors $\Delta_0^{(a)}$ by calculating a lower bound on the dimension of the vector space $V := H^0(X, \wedge^i M_{\omega_X \otimes \eta} \otimes \omega_X \otimes \eta^{-1})$ for a general curve $[X, \eta] \in \Delta_0^{(a)}$. The result does not depend on a.

Let $X = C \cup_{p,q} E$ where $E \cong \mathbb{P}^1$ is exceptional. Observe that $\omega_X|_E = \mathcal{O}_E$ while $\omega_X|_C = K_C(p+q)$. One then calculates that

$$M_{\omega_X \otimes \eta}|_C = M_{K_C(p+q) \otimes \eta_C} \text{ and } M_{\omega_X \otimes \eta}|_E = \mathcal{O}_E(-1) \oplus \mathcal{O}_E^{\oplus(g-3)}$$

We let $M := M_{K_C(p+q)\otimes\eta_C}$. By the Mayer–Vietoris sequence V is the kernel of

$$H^0\left(C, \wedge^i M \otimes K_C(p+q) \otimes \eta_C^{-1}\right) \oplus H^0\left(E, \wedge^i(\mathcal{O}_E(-1) \oplus \mathcal{O}_E^{\oplus(g-3)}) \otimes \mathcal{O}_E(-1)\right) \\ \to H^0\left(\wedge^i M_{\omega_X \otimes \eta} \otimes \omega_X \otimes \eta^{-1}|_{p+q}\right)$$

Since $\wedge^i M_{\omega_X \otimes \eta}$ has rank $\binom{2i}{i}$, the latter space has dimension $2 \cdot \binom{2i}{i}$, while the bundle on E has no sections. Using Riemann–Roch, we calculate

$$\begin{aligned} h^0\left(C, \wedge^i M \otimes K_C(p+q) \otimes \eta_C^{-1}\right) \geqslant & -(2g-3)\binom{2i-1}{i-1} \\ & + \binom{2i}{i}(2g-1+2-g) \\ = & \, 5\binom{2i-1}{i-1} \end{aligned}$$

hence the kernel has dimension at least

$$5\binom{2i-1}{i-1} - 2\binom{2i}{i} = \binom{2i-1}{i-1}$$

and therefore χ is degenerate to this order on the boundary Δ_0^{ram}. We have proved:

Proposition 3.23. *The divisor class* $[\overline{\mathcal{D}}_{g,\ell}]^{\mathrm{virt}} - \binom{2i-1}{i-1}\sum_{a=1}^{\lfloor \ell/2 \rfloor} \delta_0^{(a)}$ *is effective.*

Example 3.24. For $g = 8$ (i.e. $i = 3$) and $\ell = 3$ we obtain that

$$[\overline{\mathcal{D}}_{8,3}] - 10\delta_0^{(1)} \equiv 38\lambda - 6(\delta_0' + \delta_0'') - \frac{32}{3}\delta_0^{(1)}$$

is effective.

3.4.2 Degeneracy of $\overline{\mathcal{B}}_{8,3}$ on the boundary

Recall that our strategy to calculate the divisor class of $\overline{\mathcal{B}}_{8,3}$ was to globalize the map

$$H^0(C, L \otimes \eta) \otimes \left(\frac{H^0(C, L^{\otimes 2})}{\operatorname{Sym}^2 H^0(C, L)}\right)^{\vee} \to H^0(C, L \otimes \eta^{-1})^{\vee}$$

where $L \in W_9^3(C)$. Using Tensor-Hom adjunction, this map fails to be injective if and only if the bilinear map corresponding to the multiplication map

$$\mu_{[C,\eta,L]}\colon H^0(C, L \otimes \eta) \otimes H^0(C, L \otimes \eta^{-1}) \to H^0(C, L^{\otimes 2}) / \operatorname{Sym}^2 H^0(C, L) \quad (3.7)$$

is degenerate. We will in fact show that for general $[X, \eta] \in \Delta_0^{(1)}$ and $L \in W_9^3(X)$ the map $\mu_{[X,\eta,L]}$ is the zero map, i.e., the image of

$$H^0(X, L \otimes \eta) \otimes H^0(X, L \otimes \eta^{-1}) \to H^0(X, L^{\otimes 2})$$

is contained in the image of $\operatorname{Sym}^2 H^0(X, L)$.

A line bundle $L \in W_9^3(X)$ can be described as follows. The restriction $L_C = L|_C$ of L to C has the property $h^0(C, L_C(-p-q)) = 3$. Since L restricts to $\mathcal{O}_E$ on E, any global section $s \in H^0(X, L)$ restricts to a constant on E and hence the restriction to C has the same value at p and q. If $s|_E = 0$, then $s|_C \in H^0(C, L_C(-p-q))$. If $s|_E$ is instead a nonzero constant then $s|_C$ is a global section of L_C which vanishes neither at p nor at q. Fix such a section σ. Then we have an isomorphism

$$H^0(X, L) \cong H^0(C, L_C(-p-q)) \oplus \langle\sigma\rangle$$

We get

$$\operatorname{Sym}^2 H^0(X, L) \cong \operatorname{Sym}^2 H^0(C, L_C(-p-q)) \oplus \langle\sigma\otimes\sigma\rangle \oplus \big(\langle\sigma\rangle \otimes H^0(C, L_C(-p-q))\big) \tag{3.8}$$

and for $[C, p, q]$ general the map

$$\operatorname{Sym}^2 H^0(X, L) \to H^0(X, L^{\otimes 2})$$

is injective with one-dimensional cokernel. On the other hand, by Riemann–Roch, we have $\dim H^0(X, L^{\otimes 2}(-p-q)) = 9$ for the space of sections vanishing at p and q. Comparing this with the expression (3.8) we see that all these sections come from $\operatorname{Sym}^2 H^0(X, L)$.

Now we consider the space $H^0(X, L \otimes \eta^{-1})$. On E the line bundle $L \otimes \eta^{-1}$ restricts to $\mathcal{O}_E(-1)$ and on C to $L_C \otimes \eta_C^{-1}$. Since $H^0(E, \mathcal{O}_E(-1)) = 0$ we have the identity

$$H^0(X, L \otimes \eta^{-1}) = H^0(C, L_C \otimes \eta_C^{-1} \otimes \mathcal{O}_C(-p-q))$$

so all sections in here vanish at p and q. This implies that the multiplication map

$$H^0(X, L \otimes \eta) \otimes H^0(X, L \otimes \eta^{-1}) \to H^0(X, L^{\otimes 2})$$

factors through $H^0(X, L^{\otimes 2}(-p-q))$, hence through the image of $\operatorname{Sym}^2 H^0(X, L)$. This means that the multiplication map $\mu_{[X,\eta,L]}$ is indeed zero. We have proved:

Proposition 3.25. *The morphism $\phi\colon \mathcal{E} \to \mathcal{F}$ of* (3.5) *between vector bundles on $\mathfrak{G}_9^{3,(3)}$ is degenerate to order* 2 *over $\Delta_0^{(1)}$. Hence $[Z_1(\phi)] - 2\delta_0^{(1)}$ is effective and therefore*

$$[\overline{\mathcal{B}}_{8,3}] - 28\delta_0^{(1)} = 196\lambda - 28(\delta_0' + 2\delta_0'') - \frac{308}{3}\delta_0^{(1)}$$

is effective as well.

Theorem 3.26. *$\overline{\mathcal{R}}_{8,3}$ is of general type.*

Proof. We take the effective linear combination

$$\begin{aligned}\frac{1}{119}([\overline{\mathcal{B}}_{8,3}] - 28\delta_0^{(1)}) + \frac{5}{17}([\overline{\mathcal{D}}_{8,3}] - 10\delta_0^{(1)}) &\leqslant \frac{218}{17}\lambda - 2(\delta_0' + \delta_0'') - 4\delta_0^{(1)} \\ &= K_{\mathcal{R}'_{8,3}} - \frac{3}{17}\lambda\end{aligned}$$

hence $K_{\mathcal{R}'_{8,3}}$ is big. Now we invoke Lemma 1.26 to show that the same holds for $K_{\overline{\mathcal{R}}_{8,3}}$. ■

CHAPTER 4

Normal bundles of canonical curves

In this chapter we investigate the stability of the normal bundles of canonically embedded curves in genus 8 and 9. As was discussed in section 1.12, the normal bundle of a general canonical curve of genus $g \geqslant 7$ is conjectured to be stable (see [AFO16, Conjecture 0.4]). So far this conjecture has only been proven for $g = 7$ (loc. cit., Theorem 0.2). Here we present a complete solution to the genus 8 case:

Theorem 4.1. *The normal bundle of a canonical curve* C *of genus* 8 *is stable if and only if the curve does not have a* $\mathfrak{g}^2_7$*. Furthermore, it is polystable if and only if the curve is not tetragonal.*

To better understand the result, note that every tetragonal genus 8 curve has a $\mathfrak{g}^2_7$ ([Muk93, Lemma 3.8]).

In the proof we use S. Mukai's description of general canonical genus 8 curves as linear sections of a Grassmannian $G(2,6) \subseteq \mathbb{P}^{14}$ in an essential way. This description lets us write the normal bundle of a general curve in terms of the restriction of the tautological bundle of the Grassmannian, which gives us enough information to prove stability.

Note that the interpretation of the normal bundle in terms of Mukai bundles was also used in [AFO16] for the genus 7 case. In the hope of continuing this pattern, we study the normal bundle of a canonical genus 9 curve C in section 4.2. From the embedding of C into the symplectic Grassmannian $SpG(3,6)$ we again obtain a description of the normal bundle in terms of the tautological bundle on $SpG(3,6)$, albeit this time in a more indirect way. Currently this does not lead to a proof of stability, but we are not too far away. To quantify our results, we introduce the following notion:

Definition 4.2. Let E be a vector bundle on a curve C. Let $1 \leqslant \nu \leqslant \mathrm{rk}(E) - 1$ be an integer and let

$$m_\nu(E) = \max\{\deg(F) \mid \mathrm{rk}(F) = \nu, F \subseteq E\}$$

be the maximal degree of a rank ν subbundle of E. Then the ν*-th stability degree of* E is the number

$$s_\nu(E) = \nu \cdot \deg(E) - \mathrm{rk}(E) \cdot m_\nu(E)$$

A vector bundle E has no destabilizing subbundles of rank ν if and only if $s_\nu > 0$. In particular, E is stable if and only if $s_\nu > 0$ for all ν.

Theorem 4.3. *The stability degrees of the twisted conormal bundle* $\mathcal{N} = \mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$ *of a general canonical genus* 9 *curve are bounded as follows:*

$$\begin{aligned} s_1(\mathcal{N}) &= 36 \\ s_2(\mathcal{N}) &\geqslant 9 \\ s_3(\mathcal{N}) &\geqslant -18 \\ s_4(\mathcal{N}) &\geqslant -38 \\ s_5(\mathcal{N}) &\geqslant -44 \\ s_6(\mathcal{N}) &\geqslant -8 \end{aligned}$$

In particular, $\mathcal{N}$ *is stable with respect to subbundles of ranks* 1 *and* 2*. Additionally, if* $\mathcal{N}$ *has no* $\mathfrak{g}^2_8$ *quotient line bundle, then* $s_6(\mathcal{N}) \geqslant 6$*.*

There is an interesting connection between the description of $\mathcal{N}_{C/\mathbb{P}^8}$ in terms of the Mukai bundle and a question asked by C. Ciliberto and R. Miranda in [CM90]. On a general curve of genus 9 the Wahl map

$$\bigwedge^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 3})$$

is not of maximal rank (the only other genus where this happens is 11). In fact, its kernel is 1-dimensional and this gives rise to a line bundle $\mathscr{L}$ on (an open subset of) $\mathcal{M}_9$. We observe that a related bundle appears in an exact sequence for the normal bundle and are able to calculate the class of a globalized version over $\mathcal{M}_9$.

Being already in the Mukai bundle spirit, we also give another proof for the instability of the normal bundle of genus 6 curves in section 4.3. This proof is only slightly different from the one in [AFO16, Proposition 3.2], but fits nicely into our unified treatment of low genus curves using the geometry of Mukai's Grassmannian embeddings.

4.1 The normal bundle of canonical genus 8 curves

Recall from section 1.11 that the general genus 8 curve is a transversal linear section of a Grassmannian $G = G(2,6)$ in its Plücker embedding. The normal bundle of C then arises as the pullback of the normal bundle of G, which has a description in terms of the tautological bundle S on G. More precisely, we have $\mathcal{N}_{C/\mathbb{P}^7}(-1) = \bigwedge^2 S_C^\vee$, where S_C is the restriction of S to C.

Our first step is to prove that S_C is stable. Since exterior powers of stable bundles are polystable, we are then left with proving that $\mathcal{N}_{C/\mathbb{P}^7}$ does not decompose. This we do by direct computation.

If $W^2_7(C) \neq \emptyset$ but the curve is not tetragonal, we show that the normal bundle is not stable, but still polystable. The essential ingredient is the existence of complete intersection models of these curves in $\mathbb{P}^2 \times \mathbb{P}^2$, constructed by M. Ide and S. Mukai in [IM03].

4.1.1 Stability of the normal bundle for general curves

Description of the normal bundle

Throughout the rest of this section we will assume that C is a curve of genus $g = 8$, canonically embedded in $\mathbb{P}^7$, such that $W^2_7(C) = \emptyset$, i.e., C is Brill–Noether general. By the work of Mukai ([Muk93]), such a curve C is a transversal linear section of a Grassmannian $G = G(2,6)$, embedded by the Plücker embedding in $\mathbb{P}^{14}$. In other words, there is a 7-plane $\mathbb{P}^7 \subset \mathbb{P}^{14}$ such that $C = G \cap \mathbb{P}^7$.

The inclusion $C \hookrightarrow G$ is induced by the global sections of the *Mukai bundle* E_C on C, an up to isomorphism uniquely defined stable rank 2 bundle with $h^0(C, E_C) = 6$. If ζ is any $\mathfrak{g}^1_5$ on C and $\eta = \omega_C \otimes \zeta^{-1} \in W^3_9(C)$ its Serre dual then E_C sits in the exact sequence

$$0 \to \zeta \to E_C \to \eta \to 0$$

which is split on global sections.

Using the Euler exact sequence and the normal bundle exact sequence we calculate that for any canonically embedded curve

$$\det(\mathcal{N}_{C/\mathbb{P}^{g-1}}) = \omega_C^{\otimes(g+1)}, \quad \deg(\mathcal{N}_{C/\mathbb{P}^{g-1}}) = (g+1)(2g-2)$$

and we remark that $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ is of rank $g-2$. In the present case of genus $g = 8$ we consider the twist $\mathcal{N}_{C/\mathbb{P}^7}(-1)$ with $\det \mathcal{N}_{C/\mathbb{P}^7}(-1) = \omega_C^{\otimes 3}$ and slope $\mu(\mathcal{N}_{C/\mathbb{P}^7}(-1)) = 7$.

Because $C = \mathbb{P}^7 \cap G$, we have the split exact sequence

$$0 \to \mathcal{N}_{C/G} \to \mathcal{N}_{C/\mathbb{P}^{14}} \to \mathcal{N}_{C/\mathbb{P}^7} \to 0$$

and the exact sequence of normal bundles induced by the sequence of inclusions $C \hookrightarrow G \hookrightarrow \mathbb{P}^{14}$,

$$0 \to \mathcal{N}_{C/G} \to \mathcal{N}_{C/\mathbb{P}^{14}} \to \mathcal{N}_{G/\mathbb{P}^{14}}|_C \to 0$$

which together imply $\mathcal{N}_{C/\mathbb{P}^7} = \mathcal{N}_{G/\mathbb{P}^{14}}|_C$. But the normal bundle of a $G(2,n)$ in its Plücker embedding is explicitly given by

$$\mathcal{N}_{G(2,n)/\mathbb{P}^N} = \bigwedge^2 S^\vee \otimes \bigwedge^2 Q = \bigwedge^2 S^\vee \otimes \mathcal{O}_{G(2,n)}(1)$$

Here S is the tautological bundle over $G(2,n)$ and Q the universal quotient bundle of rank 2, sitting in the exact sequence $0 \to S \to \mathcal{O}_{G(2,n)}^{\oplus n} \to Q \to 0$. A proof of this classical fact can be found e.g., in [Man98]. Hence if S_C denotes the restriction of S to C then

$$\mathcal{N}_{C/\mathbb{P}^7}(-1) = \bigwedge^2 S_C^\vee$$

Remark 4.4. Several things are special about the case of genus 8. In fact, this is the only genus $g \geqslant 6$ where the slope of the normal bundle is an integer. This happens if and only if

$$(g-2) \mid 2(g-1)(g+1)$$

Since $(g-2)$ and $(g-1)$ have no common factors, we must have $(g-2) \mid 2(g+1)$. This in turn implies that if $a \mid (g-2)$ then $a \in \{1,2,3\}$. Equivalently, $(g-2)$ divides 6. So the only possibilities are $g = 3,4,5,8$.

By the above explicit description of $\mathcal{N}_{C/\mathbb{P}^7}$ we also see that the normal bundle of a general canonical genus 8 curve is self-dual up to twist, since for any vector bundle $\mathcal{F}$ of rank 4 we have the duality

$$\bigwedge^2 \mathcal{F} = \left(\bigwedge^2 \mathcal{F}\right)^\vee \otimes \det \mathcal{F}$$

and hence

$$\mathcal{N}_{C/\mathbb{P}^7}^\vee(2) \cong \mathcal{N}_{C/\mathbb{P}^7}(-1)$$

By comparing the degrees of $\mathcal{N}_{C/\mathbb{P}^{g-1}}(k)$ and $\mathcal{N}_{C/\mathbb{P}^{g-1}}^\vee(l)$ for $k,l \in \mathbb{Z}$ and arbitrary genus, we see that $g = 8$ is the only case $g \geqslant 6$ where this form of self-duality is possible.

Our strategy is then to derive the stability of $\mathcal{N}_{C/\mathbb{P}^7}(-1)$ from the stability of S_C. Hence we first have to understand the bundle S_C better. First of all we note some basic numerical facts. The slope of S_C is $\mu(S_C^\vee) = 7/2$. We can restrict the universal exact sequence over the Grassmannian $G(2,6)$ to C and obtain

$$0 \to S_C \to \mathcal{O}_C^{\oplus 6} \to E_C \to 0$$

This implies that $S_C^\vee$ is globally generated and $h^0(C, S_C^\vee) \geqslant 6$. We also have $\det(S_C^\vee) = \omega_C$ and $\chi(S_C^\vee) = -14$, hence $h^0(S_C^\vee) = h^1(S_C^\vee) - 14$.

Stability of the tautological bundle

We are going to prove that the restriction of the tautological bundle S of $G(2,6)$ to C is stable. The following lemma will be instrumental in what follows.

Lemma 4.5. *Let $\zeta \in W^1_5(C)$. Then there is a unique surjection $S_C^\vee \to \zeta$. This gives rise to an exact sequence*

$$0 \to Q_\eta \to S_C^\vee \to \zeta \to 0$$

where Q_η is the dual of the kernel bundle of $\eta = \omega_C \otimes \zeta^{-1}$.

Proof. The space $H^0(C, E_C \otimes \zeta^{-1}) = H^1(C, E_C \otimes \zeta)^\vee$ is easily seen to be 1-dimensional (for instance by [Muk93, Lemma 3.10]), hence $h^0(C, E_C \otimes \zeta) = 11$. Let $V = H^0(C, E_C)$. Then by tensoring

$$0 \to S_C \to V \otimes \mathcal{O}_C \to E_C \to 0$$

with ζ and taking cohomology we obtain

$$0 \to H^0(C, S_C \otimes \zeta) \to V \otimes H^0(C, \zeta) \to H^0(C, E_C \otimes \zeta)$$

We want to show that the kernel $H^0(C, S_C \otimes \zeta) = \mathrm{Hom}(S_C^\vee, \zeta)$ of the multiplication map $\mu\colon V \otimes H^0(C, \zeta) \to H^0(C, E_C \otimes \zeta)$ is one-dimensional. Using the base point free pencil trick we write

$$0 \to \zeta^{-1} \to H^0(C, \zeta) \otimes \mathcal{O}_C \to \zeta \to 0$$

and tensor this by E_C. Taking cohomology, we see that the kernel of μ is exactly $H^0(C, E_C \otimes \zeta^{-1})$, i.e., $\dim(\ker(\mu)) = 1$.

This means there is a unique nonzero morphism $S_C^\vee \to \zeta$ which surjects onto a line bundle of degree $d \leqslant \deg(\zeta) = 5$. This line bundle, as a quotient of $S_C^\vee$, must be globally generated and since $W^1_4(C) = \emptyset$ it must be ζ itself. We get an exact sequence

$$0 \to F \to S_C^\vee \to \zeta \to 0$$

where F is a rank 3 vector bundle with $\det(F) = \eta$, the Serre dual of ζ. Because $H^0(C, E_C) = H^0(C, \zeta) \oplus H^0(C, \eta)$ we get the following commutative diagram with exact columns and the upper two rows also exact:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \zeta^{-1} \arrow[r] \arrow[d] & H^0(C,\zeta)\otimes\mathcal{O}_C \arrow[r] \arrow[d] & \zeta \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_C \arrow[r] \arrow[d] & H^0(C,E_C)\otimes\mathcal{O}_C \arrow[r] \arrow[d] & E_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & ? \arrow[r] \arrow[d] & H^0(C,\eta)\otimes\mathcal{O}_C \arrow[r] \arrow[d] & \eta \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

The Snake lemma implies that the last row is exact as well, i.e., the question mark is actually M_η. ■

Remark 4.6. This shows in particular that $h^0(C, S_C^\vee) = 6$ since from the exact sequence $0 \to Q_\eta \to S_C^\vee \to \zeta \to 0$ we get the bound

$$h^0(C, S_C^\vee) \leqslant h^0(C,\zeta) + h^0(C, Q_\eta) = 2 + 4$$

and we already knew $h^0(C, S_C^\vee) \geqslant 6$.

The bundle Q_η that appears in Lemma 4.5 plays an important role in understanding the stability of $S_C^\vee$. It is not too hard to show that Q_η is itself stable.

Lemma 4.7. *Let $\eta \in W_9^3(C)$ and let Q_η be the dual of the kernel bundle of η. Then Q_η is stable and $H^0(C, Q_\eta) = H^0(C,\eta)^\vee$.*

Proof. The stability is proved in [ES12].

Let D be the intersection of C and a quadrisecant line in $\mathbb{P}^3$ where C is embedded by $|\eta|$. Then $\eta(-D) = \xi \in W_5^1(C)$. We have a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \eta^{-1} \arrow[r] \arrow[d, hook] & H^0(C,\eta)^\vee\otimes\mathcal{O}_C \arrow[r] \arrow[d] & Q_\eta \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \xi^{-1} \arrow[r] & H^0(C,\xi)^\vee\otimes\mathcal{O}_C \arrow[r] & \xi \arrow[r] & 0
\end{tikzcd}
$$

and an exact sequence

$$0 \to G \to Q_\eta \to \xi \to 0$$

where G is a rank 2 bundle with $\det(G) = \mathcal{O}_C(D)$. Since $h^0(C, Q_\eta) \geqslant 4$ we have $h^0(C, G) \geqslant 2$.

Now G is the extension

$$0 \to A \to G \to B \to 0$$

of two line bundles A, B where we can assume $\deg(A) \geqslant 0$. By stability of Q_η we have $\deg(A) \leqslant 2$ and hence $2 \leqslant \deg(B) \leqslant 4$. In any case we have $h^0(C, A) = h^0(C, B) = 1$ and hence $h^0(C, G) = 2$. Then we necessarily have $h^0(C, Q_\eta) = 4$. ■

The following lemma is well-known.

Lemma 4.8. *Let* G *be a globally generated vector bundle on* C *with* $H^0(C, G^\vee) = 0$. *Then* $h^0(C, \det(G)) \geqslant \mathrm{rk}(G) + 1$ *with equality if and only if* G *is the dual of the kernel bundle of* $\det(G)$.

Proof. Since G can be generated by $\mathrm{rk}(G) + 1$ sections, we have a surjection $\mathcal{O}_C^{\mathrm{rk}(G)+1} \to G \to 0$ whose kernel is $\det(G)^{-1}$. Dualizing this sequence, we get

$$0 \to G^\vee \to \mathcal{O}_C^{\mathrm{rk}(G)+1} \to \det(G) \to 0$$

and using $H^0(C, G^\vee) = 0$ the result follows. ■

This result can be improved, see for instance [PR88, Proposition 3.3, Lemma 3.9]. We are now in a position to show that the tautological bundle is stable.

Proposition 4.9. $S_C^\vee$ *is stable.*

Proof. Recall that $S_C^\vee$ is globally generated of slope $\mu(S_C^\vee) = 7/2$. Let

$$0 \to F \to S_C^\vee \to M \to 0 \tag{4.1}$$

be an exact sequence of vector bundles. We distinguish several possibilities, depending on the rank of M. In each case we have to prove $\mu(M) > \mu(S_C^\vee) = \frac{7}{2}$.

- $\mathrm{rk}(M) = 1$: Since $S_C^\vee$ is globally generated, so is M. Hence $h^0(C, M) \geqslant 2$ which implies $\mu(M) = \deg(M) \geqslant 5$ because $W_4^1(C) = \emptyset$.

- $\mathrm{rk}(M) = 2$: Again M is generated by global sections and we have $H^0(C, M^\vee) = 0$ by dualizing the exact sequence (4.1). Using Lemma 4.8 we have $h^0(C, \det M) \geqslant 3$ which implies $\deg(M) \geqslant 8$ (there is no $\mathfrak{g}_7^2$ on C). Hence $\mu(M) \geqslant 4$.

- $\mathrm{rk}(M) = 3$: Now F is a line bundle, more concretely $F = \omega_C \otimes L^{-1}$ where $L = \det(M)$. Again by Lemma 4.8 we have $h^0(C, L) \geqslant 4$. If the inequality is strict, then $\deg(M) \geqslant 11$ since C has no $\mathfrak{g}_{10}^4$. Hence $\mu(M) \geqslant \frac{11}{3} > \frac{7}{2}$ and we are done.

  So assume $h^0(C, L) = 4$. Then M is actually the dual Q_L of the kernel bundle of L. Since $h^0(C, F) + h^0(C, M) \geqslant h^0(C, S_C^\vee) = 6$ we have

$h^0(C, F) \geqslant 2$ and therefore $\deg(F) \geqslant 5$, i.e., $\deg(M) \leqslant 9$. Since C has no $\mathfrak{g}^3_8$ we must have $\deg(L) \geqslant 9$. Together this implies that $\deg(M) = 9$, i.e., $L \in W^3_9(C)$ and $F \in W^1_5(C)$. This means $S_C^\vee$ sits in the exact sequence

$$0 \to F \to S_C^\vee \to Q_L \to 0$$

But from Lemma 4.5 we also have

$$0 \to Q_L \to S_C^\vee \to F \to 0$$

So we have the composition $\varphi\colon Q_L \xrightarrow{\beta} S_C^\vee \xrightarrow{\alpha} Q_L$ which is nonzero since $\ker(\alpha) = F$. Because Q_L is stable, φ is a homothety and in particular invertible. Then $\beta \circ \varphi^{-1}$ induces a splitting $S_C^\vee = F \oplus Q_L$. But since we have

$$0 \to S_C \to \mathcal{O}_C^{\oplus 6} \to E_C \to 0$$

and E_C is nonsplit, this is a contradiction. ∎

Stability of the normal bundle

We have established that $S_C^\vee$ is stable. In order to prove the same for $\bigwedge^2 S_C^\vee$, we need some heavy machinery. Recall that a semistable vector bundle is called polystable if it is the direct sum of stable vector bundles of the same slope.

Theorem 4.10 ([Kob82; Lüb83; Don85; UY86]). *Let E be a vector bundle on a complex smooth projective variety. If E is Hermite–Einstein then it is polystable. If E is stable then it is an Hermite–Einstein bundle.*

Corollary 4.11. *If E is a stable vector bundle on a complex smooth projective variety, then $\wedge^q E$ is polystable. Analogous statements hold for* $\mathrm{Sym}^q E$ *and tensor products of stable vector bundles.*

Proof. E is Hermite–Einstein by Theorem 4.10, hence $\wedge^q E$ is as well (see e.g. [Lüb83]). The result then follows by using Theorem 4.10 again. ∎

Hence we know $\mathcal{N}_{C/\mathbb{P}^7}(-1) = \bigwedge^2 S_C^\vee$ is polystable and we are left with proving that it is indecomposable. We do this by excluding all possible ranks of bundles that could appear in a splitting.

Observe first that the sequence

$$0 \to Q_\eta \to S_C^\vee \to \zeta \to 0$$

from Lemma 4.5 for $\zeta \in W^1_5(C)$ and η the Serre dual, leads to

$$0 \to Q_\eta^\vee \otimes \eta \to \bigwedge^2 S_C^\vee \to Q_\eta \otimes \zeta \to 0 \tag{4.2}$$

This exact sequence will be fundamental in the proof. We will also need the following two facts:

Lemma 4.12. *Let $\xi \in W^1_5(C)$. Then $H^0(C, \mathcal{N}_{C/\mathbb{P}^7}(-1) \otimes \xi^{-1}) = 0$.*

Proof. Recall that $\mathcal{N}_{C/\mathbb{P}^7}(-1) = \mathcal{N}^\vee_{C/\mathbb{P}^7}(2)$. We have the standard identification

$$H^0(C, \mathcal{N}^\vee_{C/\mathbb{P}^7}(2)) = I_2(\omega_C)$$

i.e., the global sections of $\mathcal{N}^\vee_{C/\mathbb{P}^7}(2)$ correspond to the quadrics in $\mathbb{P}^7$ containing the canonical curve. If $\mu = \omega_C \otimes \xi^{-1}$ then $H^0(C, \mathcal{N}_{C/\mathbb{P}^7}(-1) \otimes \xi^{-1}) \neq 0$ implies that the multiplication map

$$\operatorname{Sym}^2 H^0(C, \mu) \to H^0(C, \mu^{\otimes 2})$$

is not injective. But $\mu \in W^3_9(C)$ and if the image under the induced embedding were contained in a quadric surface X, then a ruling of X would induce a $\mathfrak{g}^1_4$ on C. By assumption $W^1_4(C) = \emptyset$, so this is a contradiction. ■

Lemma 4.13. *Every sub-line bundle L of Q_η has degree at most 1. Every subbundle of Q_η of rank 2 has degree at most 4.*

Proof. Consider an exact sequence

$$0 \to L \to Q_\eta \to G \to 0$$

with L a line bundle. By increasing the degree of L, if needed, we may assume that G is a vector bundle as well. Then by Lemma 4.8 we have $h^0(C, \det(G)) \geqslant 3$ and since $W^2_7(C) = \emptyset$ we must have $\deg(G) \geqslant 8$. Hence $\deg(L) \leqslant 1$.

If $0 \to G \to Q_\eta$ is a rank 2 subbundle of maximal degree then the quotient is a globally generated line bundle L, hence $\deg(L) \geqslant 5$. This in turn implies $\deg(G) \leqslant 4$. ■

Theorem 4.14. *The normal bundle of a genus 8 curve with $W^2_7(C) = \emptyset$ is stable.*

Proof. We have to show $\bigwedge^2 S^\vee_C$ does not decompose into a direct sum of stable bundles of slope $\mu(\bigwedge^2 S^\vee_C) = 7$. Assume that we can write $\bigwedge^2 S^\vee_C = F \oplus G$ where F is stable of slope $\mu(F) = 7$. We fix some notation. Let $i_F\colon F \hookrightarrow F \oplus G$ be the inclusion and denote by q the surjective map $\bigwedge^2 S^\vee_C \to Q_\eta \otimes \zeta$ from equation (4.2). Let

$$\varphi = q \circ i_F\colon F \to Q_\eta \otimes \zeta$$

We claim that $\operatorname{rk}(\varphi)$ cannot be 1.

Assume the contrary. Then $B = \operatorname{im}(\varphi)$ is a line bundle and we have exact sequences

$$0 \to \ker(\varphi) \to F \to B \to 0$$

and

$$0 \to B \to Q_\eta \otimes \zeta \to \operatorname{coker}(\varphi) \to 0$$

Lemma 4.13 implies that every sub-line bundle of $Q_\eta \otimes \zeta$ has degree at most 6, hence $\deg(B) \leqslant 6$ as well. But F is stable and has slope $\mu(F) = 7$, hence $\deg(B) \geqslant 7$ (we even have $\deg(B) \geqslant 8$ if $\mathrm{rk}(F) \geqslant 2$). This is impossible.

Now we deal with the case $\mathrm{rk}(\varphi) = 0$, i.e., $\varphi = 0$. This means that F is contained in the kernel of q, which is $Q_\eta^\vee \otimes \eta$, that is we get $0 \to F \to Q_\eta^\vee \otimes \eta$. This is impossible since $\mu(F) = 7$ and $\mu(Q_\eta^\vee \otimes \eta) = 6$, but $Q_\eta^\vee \otimes \eta$ is stable.

We now analyze all possible cases for the rank of F. By swapping F and G, if necessary, we may assume F is stable of rank 1, 2 or 3. The case of $\mathrm{rk}(F) = 1$ can already be excluded, since then $\mathrm{rk}(\varphi) \leqslant 1$.

If $\mathrm{rk}(F) = 2$ then we must have $\mathrm{rk}(\varphi) = 2$, i.e.,

$$0 \to F \to Q_\eta \otimes \zeta \to B \to 0$$

where B is a line bundle, since every rank 2 subbundle of $Q_\eta \otimes \zeta$ has slope at most 7. Tensoring by ζ^{-1} we obtain

$$0 \to F \otimes \zeta^{-1} \to Q_\eta \to B \otimes \zeta^{-1} \to 0$$

with $\deg(B \otimes \zeta^{-1}) = 5$, hence $h^0(C, B \otimes \zeta^{-1}) = 2$. Comparing this to $h^0(C, Q_\eta)$ we find $h^0(C, F \otimes \zeta^{-1}) \geqslant 2$ and hence $h^0(C, \mathcal{N}_{C/\mathbb{P}^7}(-1) \otimes \zeta^{-1}) \geqslant 2$ in contradiction of Lemma 4.12.

Finally, consider the case $\mathrm{rk}(F) = 3$. If $\mathrm{rk}(\varphi) = 2$ then we have the sequences

$$0 \to \ker(\varphi) \to F \to \mathrm{im}(\varphi) \to 0, \quad 0 \to \mathrm{im}(\varphi) \to Q_\eta \otimes \zeta \to \mathrm{coker}(\varphi) \to 0$$

which imply $\mu(\mathrm{im}(\varphi)) > 7$ and $\mu(\mathrm{im}(\varphi)) \leqslant 7$, respectively ($Q_\eta$ does not contain a rank 2 subbundle of degree at least 5). The last remaining possibility is then $\mathrm{rk}(\varphi) = 3$, i.e., $0 \to F \to Q_\eta \otimes \zeta$. The quotient is a torsion sheaf of length 3. Tensoring the sequence by ζ^{-1} and comparing global sections, we again find $h^0(C, F \otimes \zeta^{-1}) \geqslant 1$ in contradiction to $h^0(C, \mathcal{N}_{C/\mathbb{P}^7}(-1) \otimes \zeta^{-1}) = 0$. ■

4.1.2 Auxiliary results about the tautological bundle

The tautological bundle S_C on the general curve C is the kernel bundle of the Mukai bundle E_C, which enjoys certain uniqueness properties. Our aim in this section is to transfer these properties to the tautological bundle, so that it is uniquely characterized without reference to E_C. These lemmata, albeit not difficult to prove, further demonstrate some techniques in calculations involving vector bundles on curves.

We will need the following lemma of Mukai:

Lemma 4.15 ([Muk10, Lemma 1.1]). *Let ξ be a line bundle on a curve C and $\eta = \omega_C \otimes \xi^{-1}$ be the Serre adjoint. Let $m = h^0(C, \eta)$. For any vector bundle E of rank m on C we have*

$$\hom(E, \xi) + \hom(Q_\eta, E) \geqslant r(h^0(C, E) - \deg(\eta)) - \chi(E)$$

Let C be a curve of genus 8 without a $\mathfrak{g}^2_7$ and let S_C be the restriction of the tautological bundle from $G(2,6)$ to C. Let $\xi \in W^1_5(C)$ and $\eta = \omega_C \otimes \xi^{-1}$.

Lemma 4.16. *There is an up to scaling unique nontrivial extension F of ξ by Q_η with $h^0(C,F) = 6$.*

Proof. These extensions are parametrized by nontrivial elements in the cokernel of

$$H^0(C,\xi) \otimes H^0(C, M_\eta \otimes \omega_C) \to H^0(C, M_\eta \otimes \omega_C \otimes \xi)$$

The kernel of this map is $H^0(C, M_\eta \otimes \eta)$ by the base point free pencil trick. This in turn is equal to $\wedge^2 H^0(C,\eta)$ since $I_2(\eta) = 0$. Furthermore, we have $h^0(C, M_\eta \otimes \omega_C) = h^1(C, Q_\eta) = 16$ and $h^0(C, M_\eta \otimes \omega_C \otimes \xi) = 27$. Counting dimensions, we see that the cokernel is one-dimensional. ■

Lemma 4.17. *$S_C^\vee$ is the unique stable rank 4 bundle with canonical determinant and $h^0(C, S_C^\vee) \geqslant 6$.*

Proof. Let F be a stable rank 4 bundle with $\det(F) = \omega_C$ and $h^0(C,F) \geqslant 6$. Let $\xi \in W^1_5(C)$ and $\eta = \omega_C \otimes \xi^{-1}$. By Lemma 4.15 we have

$$\hom(F,\xi) + \hom(Q_\eta, F) \geqslant 4 \cdot (h^0(C,F) - 4 - 2) + 2 \geqslant 2$$

Assume first that $\hom(Q_\eta, F) \geqslant 1$. If Q_η is not a sub vector bundle, then by $\mu(F) = 3 + \frac{1}{2}$ and stability we have

$$0 \to Q \to F \to \xi(-p) \to 0$$

where Q is of degree 10 and $Q_\eta \subseteq Q$. Then we would necessarily have $h^0(C,Q) \geqslant 5$ and hence Q would be globally generated. But then its determinant, $\eta(p) \in W^3_9(C)$, would also be globally generated, which it is not. Hence we have the exact sequence

$$0 \to Q_\eta \to F \to \xi \to 0$$

Now the result follows from Lemma 4.16.

Now we assume that $\hom(F,\xi) \geqslant 1$. If the map $f \colon F \to \xi$ is not a surjection, then by stability of F it at least surjects onto $\xi(-p)$ for some p. We obtain the exact sequence

$$0 \to Q \to F \to \xi(-p) \to 0$$

where $\det(Q) = \eta(p)$ and $h^0(C,Q) \geqslant 5$. Let G be the image of the evaluation map $H^0(C,Q) \to Q$. Then G cannot be of rank 1 (there is no line bundle of degree at most 10 with that many sections) and it cannot be of rank 2 because the sequence $0 \to \mathcal{O}_C \to G \to \det(G) \to 0$ would imply $h^0(C, \det(G)) \geqslant 4$, hence $\det(G) = \eta$ and $\mu(G) = 4 + \frac{1}{2}$ contradicts stability of F. Hence G is of rank 3 and $\det(G) = \eta$. It sits in the exact sequence $0 \to \mathcal{O}_C \to G \to \eta \to 0$.

But there are no non-trivial H^0-split extensions of this type. This contradicts stability and we conclude that we have

$$0 \to Q \to F \to \xi \to 0$$

where $h^0(C, Q) \geqslant 4$. Then Q is necessarily semi-stable. Repeating the argument from before we see that necessarily $h^0(C, Q) = 4$ and the image G of $H^0(C, Q) \to Q$ is of rank 3 (in the rank 2 case the determinant of G would be a $\mathfrak{g}^2_8$, which contradicts semistability). As before, if $G \neq Q$ then $\det(G) \in W^2_8(C)$ and $G \in \operatorname{Ext}^1(\mathfrak{g}^2_8, \mathcal{O}_C)$ which contains no non-trivial H^0-split extensions. Therefore Q is globally generated and by Lemma 4.8 we have $Q = Q_\eta$. ∎

4.1.3 Polystability of the normal bundle on curves with a $\mathfrak{g}^2_7$

We now go from general genus 8 curves to slightly more special ones. Ide and Mukai describe canonical models of genus 8 curves having a $\mathfrak{g}^2_7$ in [IM03]. As long as $W^1_4(C) = \emptyset$, the scheme $W^2_7(C)$ has length exactly 2. In the general case, there are precisely two non-autoresidual $\mathfrak{g}^2_7$, say $W^2_7(C) = \{\alpha, \beta\}$ with $\alpha \neq \beta$. We then get a map $\varphi\colon C \to \mathbb{P}^2 \times \mathbb{P}^2$ induced by the linear system $|\alpha| \times |\beta|$. It is shown in [IM03] that φ is an embedding and its image a complete intersection of divisors of type $(1, 2)$, $(2, 1)$ and $(1, 1)$.

Let $\mathcal{S}$ be the image of the Segre embedding $\sigma\colon \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$. Then the canonical model of C lies in $\sigma(\mathbb{P}^2 \times \mathbb{P}^2)$ intersected with a hyperplane $H = \mathbb{P}^7$. We let $W = \mathcal{S} \cap H$. So inside W, C is the transversal intersection of a divisor $D_{1,2}$ of type $(1, 2)$ and one divisor $D_{2,1}$ of type $(2, 1)$. We have the exact sequence

$$0 \to \mathcal{N}_{C/W} \to \mathcal{N}_{C/\mathbb{P}^7} \to \mathcal{N}_{W/\mathbb{P}^7}|_C \to 0$$

Since C is a complete intersection in W, the normal bundle is just

$$\mathcal{N}_{C/W} = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 2)|_C \oplus \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(2, 1)|_C = \omega_C \otimes (\alpha \oplus \beta)$$

or equivalently $\mathcal{N}_{C/W}(-1) = \alpha \oplus \beta$ under the canonical embedding with $\mathcal{O}_C(1) = \omega_C$. We immediately get that $\mathcal{N}_{C/\mathbb{P}^7}(-1)$ is not stable, since it contains $\alpha \oplus \beta$ as a subbundle.

The following fact seems to be well-known and follows from the more general description of tangent bundles of flag varieties:

Lemma 4.18. *The normal bundle of $\mathcal{S}$ in $\mathbb{P}^8$ is*

$$K_{(1,0)} \otimes K_{(0,1)} \otimes \mathcal{O}_{\mathcal{S}}(1, 1)$$

where $K_{(i,j)}$ is the kernel of the evaluation map $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(i, j)) \otimes \mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{S}}(i, j)$.

Proof. Consider $\mathbb{P}(V)$ and $\mathbb{P}(W)$ and $\mathbb{P}(V \otimes W)$. If L is the tautological bundle on projective space then

$$\mathcal{N}_{\mathbb{P}(V) \times \mathbb{P}(W)/\mathbb{P}(V \otimes W)} = \mathcal{H}om(L, V/L) \otimes \mathcal{H}om(L, W/L)$$

This is explained, for instance, in [Man15, Section 4.1]. Now the result follows from expanding this expression. ∎

Using $\mathcal{N}_{W/H} = \mathcal{N}_{\mathcal{S}/\mathbb{P}^8}|_W$ and pulling this back to C we obtain that the twist $\mathcal{N}_{W/\mathbb{P}^7}|_C(-1)$ is equal to $Q_\alpha \otimes Q_\beta$. Here Q_α and Q_β are the duals of the kernel bundles associated to α and β, respectively. Hence $\mathcal{N}_{C/\mathbb{P}^7}(-1)$ sits in the exact sequence

$$0 \to \alpha \oplus \beta \to \mathcal{N}_{C/\mathbb{P}^7}(-1) \to Q_\alpha \otimes Q_\beta \to 0$$

and because $\mathcal{N}_{C/\mathbb{P}^7}$ is self-dual up to twist, this sequence splits. Therefore

$$\mathcal{N}_{C/\mathbb{P}^7}(-1) = \alpha \oplus (Q_\alpha \otimes Q_\beta) \oplus \beta \tag{4.3}$$

To prove that this is in fact polystable, we will show the stability of $Q_\alpha \otimes Q_\beta$. Some preliminary results will be needed.

Lemma 4.19. Q_β *is stable and* $h^0(C, Q_\beta) = 3$. *The maximal degree of a line subbundle of* Q_β *is* 2. *Analogous statements hold of course also for* Q_α.

Proof. Stability is shown as in Lemma 4.7. Now Q_β sits in the exact sequence

$$0 \to \mathcal{O}_C(a+b) \to Q_\beta \to \beta \otimes \mathcal{O}_C(-a-b) \to 0$$

where $a, b \in C$ are map to a singular point of the degree 7 plane model induced by β, hence $\beta \otimes \mathcal{O}_C(-a-b)$ is a $\mathfrak{g}^1_5$. This shows $h^0(C, Q_\beta) = 3$ and $\deg(L) \leqslant 2$ for every line subbundle L of Q_β. ∎

Remark 4.20. Since $h^0(C, \mathcal{N}_{C/\mathbb{P}^7}) = 15$, eq. (4.3) yields $h^0(C, Q_\alpha \otimes Q_\beta) = 9$ as a consequence. One can also proceed similarly to [Muk10, Section 5], to prove this result directly.

Lemma 4.21. $Q_\alpha \otimes Q_\beta$ *is stable.*

Proof. As a tensor product of stable bundles we already know $Q_\alpha \otimes Q_\beta$ is polystable (see Theorem 4.10) of slope $\mu(Q_\alpha \otimes Q_\beta) = 7$.

Consider the exact sequence

$$0 \to Q_\alpha(a+b) \xrightarrow{p} Q_\alpha \otimes Q_\beta \xrightarrow{q} Q_\alpha \otimes \xi \to 0$$

where $\beta(-a-b) = \xi \in W^1_5(C)$ and assume there exists a line bundle direct summand L of $Q_\alpha \otimes Q_\beta$. The induced map $L \to Q_\alpha \otimes \xi$ cannot be zero, since otherwise we get $L \hookrightarrow \ker(q) = Q_\alpha(a+b)$, contradicting the stability of $Q_\alpha(a+b)$. So $L \to Q_\alpha \otimes \xi$ is injective, hence we obtain

$$0 \to L \to Q_\alpha \otimes \xi \to B \to 0$$

where B is a line bundle of degree 10. Counting global sections we obtain $h^0(C, L \otimes \xi^{-1}) = 1$ and hence ξ is a subbundle of L. But L, as a direct summand of the globally generated bundle $Q_\alpha \otimes Q_\beta$, is itself globally generated, so $L \in W^2_7(C)$. If $L = \alpha$ then

$$H^0(C, Q_\alpha \otimes Q_\beta \otimes L^{-1}) = H^0(C, Q_\alpha^\vee \otimes Q_\beta) = \mathrm{Hom}(Q_\alpha, Q_\beta) = 0$$

by the stability of Q_α and Q_β and the fact that Q_α and Q_β are not isomorphic. The same statement holds if $L = \beta$. So $Q_\alpha \otimes Q_\beta$ does not contain a line bundle as a direct summand.

Assume now that $Q_\alpha \otimes Q_\beta = F \oplus G$ where F and G are two rank 2 bundles. Consider again the sequence

$$0 \to Q_\alpha(a+b) \xrightarrow{p} F \oplus G \xrightarrow{q} Q_\alpha \otimes \xi \to 0$$

and the induced maps $\varphi\colon Q_\alpha(a+b) \to F$ and $\psi\colon Q_\alpha(a+b) \to G$. If $\varphi = 0$ this would induce a surjection $Q_\alpha \otimes \xi \to F \to 0$, which is impossible. If the rank of φ was 1 then we would get

$$Q_\alpha(a+b) \to \mathrm{im}(\varphi) \to 0, \quad 0 \to \mathrm{im}(\varphi) \to F$$

and hence $\deg(\mathrm{im}(\varphi)) \geqslant 7$ and $\deg(\mathrm{im}(\varphi)) < 7$, respectively. So the only possibility is that φ is injective. The same reasoning applies to ψ, hence we obtain injections $Q_\alpha(a+b) \hookrightarrow F$ and $Q_\alpha(a+b) \hookrightarrow G$. This implies $h^0(C, F(-a)) \geqslant 3$, $h^0(C, G(-a)) \geqslant 3$, i.e., $h^0(C, Q_\alpha \otimes Q_\beta \otimes \mathcal{O}_C(-a)) \geqslant 6$. But $Q_\alpha \otimes Q_\beta$ is globally generated of rank 4, so

$$h^0(C, Q_\alpha \otimes Q_\beta \otimes \mathcal{O}_C(-a)) = h^0(C, Q_\alpha \otimes Q_\beta) - 4 = 5$$

which is again a contradiction. ∎

Remark 4.22. Since we have $E_C = \alpha \oplus \beta$ for the Mukai bundle on curves with a $\mathfrak{g}^2_7$ but no $\mathfrak{g}^1_4$ we can still form the exact sequence

$$0 \to S_C \to H^0(C, E_C) \otimes \mathcal{O}_C \to E_C \to 0$$

even though it does not come from a tautological exact sequence over some Grassmannian. Then $S_C^\vee = Q_\alpha \oplus Q_\beta$. We also get

$$\bigwedge^2 S_C^\vee = \alpha \oplus (Q_\alpha \otimes Q_\beta) \oplus \beta$$

which is exactly the normal bundle $\mathcal{N}_{C/\mathbb{P}^7}(-1)$ of the canonical embedding of C. So the normal bundle has intrinsically the same description (in terms of the Mukai bundle) as on the general curve.

4.2 The normal bundle of canonical genus 9 curves

The set-up for genus 9 is slightly more complicated than in genus 8. As is proved in [Muk10], the Mukai bundle E_C on a general (i.e. non-pentagonal) curve is of rank 3 and induces an embedding $C \to \mathrm{SpG}(3,6) \subseteq \mathbb{P}^{13}$ into the Grassmannian $\mathrm{SpG}(3,6)$ of Lagrangian subspaces of $H^0(C, E_C)$. The curve C in its canonical embedding is then recovered as the transversal intersection of $\mathrm{SpG}(3,6)$ with an 8-plane. We use this description in what follows to obtain more information about the normal bundle of the canonical embedding of $C \subseteq \mathbb{P}^8$.

4.2.1 Description of the normal bundle

The symplectic Grassmannian $\mathrm{SpG}(3,6)$ is a subvariety of the Grassmannian $G = G(3,6)$ of 3-planes in $H^0(C, E_C)$. On G we have the universal exact sequence

$$0 \to S_G \to V \otimes \mathcal{O}_G \to Q_G \to 0$$

where Q_G is the universal quotient bundle and S_G is the tautological bundle. The sequence restricts to $X = \mathrm{SpG}(3,6)$ and on X we can identify S_X with $Q_X^\vee$:

$$0 \to Q_X^\vee \to V \otimes \mathcal{O}_X \to Q_X \to 0$$

X is of dimension 6 and embedded in $\mathbb{P}^{13}$ by a restricted Plücker embedding. Its normal bundle is therefore of rank 7.

Lemma 4.23 ([Muk10, Section 2]). *The normal bundle of* $\mathrm{SpG}(3,6) \subseteq \mathbb{P}^{13}$ *sits in the exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{N}^\vee_{X/\mathbb{P}^{13}}(2) \to \mathrm{Sym}^2 Q_X \to 0$$

where $\mathcal{O}_X(1) = \det Q_X$.

From the equality $C = \mathbb{P}^8 \cap \mathrm{SpG}(3,6)$ it follows that the normal bundle $\mathcal{N}_{C/\mathbb{P}^8}$ of the canonical embedding is equal to the restriction to C of the normal bundle $\mathcal{N}_{\mathrm{SpG}(3,6)/\mathbb{P}^{13}}$ of $\mathrm{SpG}(3,6)$. Since the restriction of Q_X to C is the Mukai bundle E_C, we obtain the following exact sequence for the normal bundle of C.

Lemma 4.24. *The twisted conormal bundle of* $C \subseteq \mathbb{P}^8$ *in its canonical embedding sits in the exact sequence*

$$0 \to \mathcal{O}_C \to \mathcal{N}^\vee_{C/\mathbb{P}^8}(2) \to \mathrm{Sym}^2 E_C \to 0 \tag{4.4}$$

Observe that the universal exact sequence of X restricts to

$$0 \to E_C^\vee \to H^0(C, E_C) \to E_C \to 0 \tag{4.5}$$

on C.

4.2.2 Facts about the Mukai bundle

We first study the Mukai bundle E_C more closely. In particular, we would like to understand its stability properties. Some properties of E_C are proved in [Muk10] and we quickly summarize them.

A basic result that we will need is the following:

Lemma 4.25 ([LN10]). *Mercat's conjecture holds for rank* 2 *bundles on all genus* 9 *curves. In particular, for a semistable rank* 2 *bundle* F *with* $h^0(C, F) \geqslant 4$ *on a general curve* C *of genus* 9 *we have*

$$\deg(F) \geqslant 4 + 2 \cdot h^0(C, F)$$

To construct E_C without reference to SpG(3, 6), start from any $\alpha \in W^2_8(C)$. Because C is general and $\rho(9, 2, 8) = 0$, there are finitely many such $\mathfrak{g}^2_8$ on C. The Serre adjoint $\beta = \omega_C \otimes \alpha^{-1}$ is a $\mathfrak{g}^2_8$ as well. On the general curve we have $\alpha \neq \beta$ and both line bundles induce a birational map to a degree 8 model of C in $\mathbb{P}^2$ with only nodes as singularities.

Consider Q_β, the dual of the kernel bundle of β. Then Q_β is stable and $h^0(C, Q_\beta) = 3$. We can characterize E_C as the unique extension

$$0 \to Q_\beta \to E_C \to \alpha \to 0 \tag{4.6}$$

of α by Q_β which is split on global sections. This also implies immediately that E_C is globally generated. The nonexistence of a $\mathfrak{g}^1_5$ on C is used to prove that E_C is stable:

Lemma 4.26 ([Muk10, Remark 5.7]). E_C *is stable.*

The following two lemmata further improve this result:

Lemma 4.27. $\mathrm{Hom}(A, E_C) = 0$ *for all line bundles* L *with* $\deg(A) \geqslant 3$.

Proof. Let $0 \to A \to E_C \to Q \to 0$ be an exact sequence of vector bundles where $\deg(A)$ is maximal. In any case $\deg(A) \leqslant 5$ by the stability of E_C. Hence $h^0(C, A) \leqslant 1$ and we must have $h^0(C, Q) \geqslant 5$.

First assume that Q is semistable. Then by Lemma 4.25 we must have $\deg(Q) \geqslant 14$ and it follows that $\deg(L) \leqslant 2$.

Now if Q is not semistable then we get a sequence

$$0 \to L \to Q \to M \to 0$$

where L and M are line bundles and where $\deg(L) > \mu(Q)$. Note that, as a quotient of E_C, the bundle Q must have slope $\mu(Q) > \frac{16}{3} = \mu(E_C)$. This implies $\deg(L) \geqslant 6$. Observe that Q is globally generated, so the same holds for M. Because there are no $\mathfrak{g}^1_5$ on C, this means $\deg(M) \geqslant 6$. But since $h^0(C, Q) \geqslant 5$ at least one of L and M must have 3 independent global sections, hence must be of degree at least 8. Summing up, we must have $\deg(Q) = \deg(L) + \deg(M) \geqslant 14$. This implies $\deg(A) \leqslant 2$. ∎

Lemma 4.28. $\mathrm{Hom}(F, E_C) = 0$ *for all rank* 2 *bundles* F *with* $\mu(F) > 4$.

Proof. Let $0 \to F \to E_C \to L \to 0$ be a sequence of vector bundles and assume $\mu(F) > 4$, i.e., $\deg(F) \geqslant 9$. Then $6 \leqslant \deg(L) \leqslant 7$ since L must be globally generated. From the nonexistence of a $\mathfrak{g}^3_7$ we obtain $h^0(C, L) = 2$, which in turn implies $h^0(C, F) \geqslant 4$. By Lemma 4.25 the bundle F must be unstable. Then we have a subbundle $0 \to A \to F$ with $\deg(A) \geqslant 5$ which will also be a subbundle of E_C, a contradiction to Lemma 4.27. ∎

Corollary 4.29. *All quotient line bundles* L *of* E_C *have* $\deg(L) \geqslant 8$. *All rank* 2 *quotient bundles* Q *of* E_C *have* $\mu(Q) \geqslant 7$.

We remark that the bounds above are sharp. The rank two bundle Q_β is a subbundle of E_C with $\mu(Q_\beta) = 4$. Furthermore, Q_β contains a sub-line bundle of degree 2, which makes for a very useful result on its own:

Lemma 4.30. *Let* $p, q \in C$ *be two points mapping to a node under the map induced by* β. *Then we have an exact sequence*

$$0 \to \mathcal{O}_C(p+q) \to Q_\beta \to \xi \to 0$$

where $\xi = \beta(-p-q) \in W^1_6(C)$ *is globally generated.*

Proof. Apply Lemma 1.49 to the base point free subbundle $\xi \subseteq \alpha$. ∎

4.2.3 Further results about global sections and stability

In this section we obtain some purely technical results on the number of global sections and stability properties of various vector bundles on C.

Lemma 4.31. *We have*

$$h^0(C, Q_\beta^\vee \otimes \omega) = h^0(C, Q_\beta \otimes \alpha) = 11$$

hence the map

$$H^0(C, \beta) \otimes H^0(C, \omega_C) \to H^0(C, \omega_C \otimes \beta)$$

is surjective.

Proof. Dualize the sequence

$$0 \to \mathcal{O}_C(p+q) \to Q_\beta \to \xi \to 0$$

and tensor by ω_C to obtain

$$0 \to \omega_C \otimes \xi^{-1} \to Q_\beta^\vee \otimes \omega_C \to \omega_C(-p-q) \to 0$$

The bundle $\omega_C \otimes \xi^{-1}$ is a $\mathfrak{g}^3_{10}$ and $h^0(C, \omega_C(-p-q)) = 7$. Hence we get the inequality $h^0(C, Q_\beta^\vee \otimes \omega_C) \leqslant 11$. The opposite inequality is obtained from tensoring

$$0 \to Q_\beta^\vee \to H^0(C, \beta) \otimes \mathcal{O}_C \to \beta \to 0$$

with ω_C. Then we obtain

$$0 \to H^0(C, Q_\beta^\vee \otimes \omega_C) \to H^0(C, \beta) \otimes H^0(C, \omega_C) \to H^0(C, \omega_C \otimes \beta)$$

and the kernel of the multiplication map has to be at least 11-dimensional. ∎

Remark 4.32. The stability of Q_β implies

$$h^1(C, Q_\beta \otimes Q_\beta \otimes \alpha) = h^0(C, Q_\beta \otimes Q_\beta^\vee) = \hom(Q_\beta, Q_\beta) = 1$$

and therefore $h^0(C, Q_\beta \otimes Q_\beta \otimes \alpha) = 33$. We also have

$$h^0(C, Q_\alpha \otimes Q_\beta^\vee) = \hom(Q_\beta, Q_\alpha) = 0$$

Lemma 4.33 ([Muk10, Lemma 5.2])**.** *We have* $h^0(C, Q_\alpha \otimes Q_\beta) = 10$.

Lemma 4.34. *Every quotient line bundle of* $\operatorname{Sym}^2 Q_\beta$ *has degree* $\geqslant 12$. *Every subbundle of rank* 2 *has* $\mu \leqslant 6$. *These bounds are sharp.*

Proof. Consider $\operatorname{Sym}^2 Q_\beta \to L \to 0$. Dualizing this and tensoring by $\beta^{\otimes 2}$ we get $0 \to L^{-1} \otimes \beta^{\otimes 2} \to \operatorname{Sym}^2 Q_\beta$, hence $\deg(L^{-1} \otimes \beta^{\otimes 2}) \leqslant 4$, i.e., $\deg(L) \geqslant 12$.

To see that this bound is sharp let $\xi = \beta(-a-b) \in W^1_6$. Then from the exact sequence

$$0 \to \mathcal{O}_C(a+b) \to Q_\beta \to \xi \to 0$$

we obtain the exact sequence

$$0 \to Q_\beta(a+b) \to \operatorname{Sym}^2 Q_\beta \to \xi^{\otimes 2} \to 0$$

which shows the claim. ∎

Lemma 4.35. *We have* $h^0(C, \wedge^2 E_C) = 14$.

Proof. Since E_C is of rank 3 and determinant $\det(E_C) = \omega_C$, we have the duality $E_C^\vee \otimes \omega_C = \wedge^2 E_C$. Hence Serre duality and Riemann–Roch imply

$$14 = h^1(C, E_C) = h^0(C, E_C^\vee \otimes \omega) = h^0(C, \wedge^2 E_C)$$ ∎

4.2.4 Facts about the symmetric square of the Mukai bundle

We turn our attention to $\operatorname{Sym}^2 E_C$. First we need to calculate the dimension of its space of global sections. Taking symmetric squares in the sequence (4.5) leads to

$$0 \to \wedge^2 E_C^\vee \to E_C^\vee \otimes H^0(C, E_C) \to \operatorname{Sym}^2 H^0(C, E_C) \to \operatorname{Sym}^2 E_C \to 0 \qquad (4.7)$$

which by taking global sections yields an injection

$$\operatorname{Sym}^2 H^0(C, E_C) \to H^0(C, \operatorname{Sym}^2 E_C)$$

In particular, $h^0(C, \operatorname{Sym}^2 E_C) \geqslant 21$. The following lemma proves that the above map is in fact an isomorphism.

Lemma 4.36. *We have* $\dim H^0(C, \mathrm{Sym}^2 E_C) = 21$.

Proof. From the defining exact sequence (4.6) we obtain

$$0 \to H^0(C, Q_\beta \otimes E_C) \to H^0(C, E_C \otimes E_C) \to H^0(C, E_C \otimes \alpha)$$

by tensoring with E_C and taking global sections. By Riemann–Roch we have $h^0(C, E_C \otimes \alpha) = 17$. We claim that $h^0(C, Q_\beta \otimes E_C) = 18$. Then together these results will imply $h^0(C, E_C \otimes E_C) = 35$. Since $h^0(C, \wedge^2 E_C) = 14$ by Lemma 4.35, we will be done.

It remains to prove the claim. First recall from Lemma 4.31 the dimension count $h^0(C, Q_\beta \otimes \alpha) = 11$. From the exact sequence

$$0 \to H^0(C, Q_\beta) \to H^0(C, E_C) \to H^0(C, \alpha) \to 0$$

we obtain the following exact diagram by writing down multiplication maps with $H^0(C, Q_\beta \otimes \alpha)$ and their kernels:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H^0(Q_\beta^\vee \otimes \alpha) \arrow[r] \arrow[d] & H^0(Q_\beta) \otimes H^0(Q_\beta \otimes \alpha) \arrow[r] \arrow[d] & H^0(Q_\beta \otimes Q_\beta \otimes \alpha) \arrow[d] \\
0 \arrow[r] & H^0(E_C^\vee \otimes Q_\beta \otimes \alpha) \arrow[r] \arrow[d] & H^0(E_C) \otimes H^0(Q_\beta \otimes \alpha) \arrow[r] \arrow[d] & H^0(E_C \otimes Q_\beta \otimes \alpha) \arrow[d] \\
0 \arrow[r] & H^0(Q_\alpha \otimes Q_\beta) \arrow[r] & H^0(\alpha) \otimes H^0(Q_\beta \otimes \alpha) \arrow[r] \arrow[d] & H^0(Q_\beta \otimes \alpha^{\otimes 2}) \\
 & & 0 &
\end{tikzcd}$$

Since $H^0(C, Q_\beta^\vee \otimes \alpha) = 0$, the first row remains exact when adding a 0 on the right. This implies

$$h^0(C, E_C^\vee \otimes Q_\beta \otimes \alpha) \leqslant h^0(C, Q_\alpha \otimes Q_\beta) = 10$$

and by counting dimensions in the second row we see that we actually have equality. By Riemann–Roch we have $h^0(C, E_C^\vee \otimes Q_\beta \otimes \alpha) = h^1(C, E_C \otimes Q_\beta)$ and therefore $h^0(C, E_C \otimes Q_\beta) = 18$ as claimed. ■

The stability of E_C implies by Corollary 4.11 that $\mathrm{Sym}^2 E_C$ is polystable. The exact sequence (4.6) gives rise to the filtrations

$$0 \to \mathrm{Sym}^2 Q_\beta \to \mathrm{Sym}^2 E_C \to E_C \otimes \alpha \to 0$$

and

$$0 \to E_C \otimes Q_\beta/\beta \to \mathrm{Sym}^2 E_C \to \alpha^{\otimes 2} \to 0$$

From these sequences we can deduce the following:

Lemma 4.37. *Every quotient line bundle of* $\operatorname{Sym}^2 E_C$ *has degree* $\geqslant 12$.

Proof. Consider the exact sequence

$$0 \to \operatorname{Sym}^2 Q_\beta \to \operatorname{Sym}^2 E_C \to E_C \otimes \alpha \to 0$$

From any quotient line bundle L of $\operatorname{Sym}^2 E_C$ we obtain an induced map $\varphi\colon \operatorname{Sym}^2 Q_\beta \to L$. If $\varphi = 0$, then by the universal property of cokernels we get a nonzero map $E_C \otimes \alpha \to L$, hence $\deg(L) \geqslant 16$ by Corollary 4.29. Now assume $\varphi \neq 0$. By Lemma 4.34 every quotient line bundle of $\operatorname{Sym}^2 Q_\beta$ has degree at least 12. Hence $\deg(L) \geqslant 12$ as well. ■

4.2.5 Stability properties of the normal bundle

Before we begin tackling the stability properties, we quickly discuss global sections and global generation.

Lemma 4.38. $h^0(C, \mathcal{N}^\vee_{C/\mathbb{P}^8}(2)) = 22$ *and* $\mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$ *is globally generated.*

Proof. Observe that

$$H^0(C, \mathcal{N}^\vee_{C/\mathbb{P}^8}(2)) = I_2(K_C) \oplus \ker \psi$$

where $I_2(K_C)$ is the space of quadrics containing C, and by ψ we denote the Wahl map $\wedge^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 3})$. We have $\dim I_2(K_C) = 21$ and in [CM90] it is proved that $\dim \ker \psi = 1$.

To see global generation, consider the map

$$f\colon I_2(K_C) \otimes \mathcal{O}_C \to \mathcal{N}^\vee_{C/\mathbb{P}^8}(2)|_p$$

to the fiber at a point p. Its kernel are quadrics in $\mathbb{P}^8$ vanishing on the curve, but also vanishing at p to order 2. Hence it is naturally identified with the kernel of the map

$$\operatorname{Sym}^2 H^0(C, \omega_C(-p)) \to H^0(C, \omega_C^{\otimes 2} \otimes \mathcal{O}_C(-2p))$$

Since this map is surjective ([GL86]) we can calculate the dimension of the kernel and hence the image of f. We get $\dim \operatorname{im}(f) = g-2$, so f is surjective. ■

The Wahl map will come up again later and we will need a result about its injectivity on the general curve:

Theorem 4.39 ([Far05, Theorem 1.3])**.** *For the general curve* C *of genus* g *and for any line bundle* L *on* C *of degree* $d \leqslant g+2$ *the Gaussian map* ψ_L *is injective.*

Not only is $\mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$ globally generated, its twisted dual $\mathcal{N}_{C/\mathbb{P}^8}(-1)$ is as well.

Lemma 4.40. $\mathcal{N}_{C/\mathbb{P}^8}(-1)$ *is globally generated.*

Proof. We imitate the proof in [AFO16, p. 12]. Via Serre duality, the claim is equivalent to the equality $h^0(C, \mathcal{N}^\vee_{C/\mathbb{P}^8}(2) \otimes \mathcal{O}_C(p)) = h^0(C, \mathcal{N}^\vee_{C/\mathbb{P}^8}(2)) = 22$ for every $p \in C$. Consider the pullback of the ideal sequence of C in $\mathbb{P}^8$ to C and its twist by $K_C^{\otimes 2}(p)$:

$$0 \to \mathcal{N}^\vee_{C/\mathbb{P}^8}(2) \otimes \mathcal{O}_C(p) \to M_{K_C} \otimes K_C(p) \to K_C^{\otimes 3}(p) \to 0$$

So we obtain that $H^0(C, \mathcal{N}^\vee_{C/\mathbb{P}^8}(2) \otimes \mathcal{O}_C(p))$ is the kernel of the twisted Wahl map

$$H^0(C, M_{K_C} \otimes K_C(p)) \to H^0(C, K_C^{\otimes 3}(p))$$

But because Q_{K_C} is globally generated, the dimension of $H^0(C, M_{K_C} \otimes K_C(p))$ is equal to that of $H^0(C, M_{K_C} \otimes K_C)$. ∎

For brevity, let $\mathcal{N} = \mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$ and let $p\colon \mathcal{N} \to \operatorname{Sym}^2 E_C$ be the map in (4.4). We start our analysis by noting $\mu(\mathcal{N}) = 64/7$ and $\mu(\operatorname{Sym}^2 E_C) = 32/3$. Now assume

$$0 \to F \xrightarrow{a} \mathcal{N} \to M \to 0 \tag{4.8}$$

is a destabilizing sequence of vector bundles, i.e., $\mu(F) \geqslant 9 + \frac{1}{7}$. We obtain a map $p \circ a\colon F \to \operatorname{Sym}^2 E_C$ and then $0 \to \ker(p \circ a) \to \mathcal{O}_C$ from (4.4) and the universal property of the kernel. This implies $\operatorname{rk}(\ker(p \circ a)) \leqslant 1$ and $\deg(\operatorname{im}(p \circ a)) \geqslant \deg F$. We also get

$$0 \to \ker(p \circ a) \to F \to \operatorname{im}(p \circ a) \to 0$$

and

$$0 \to \operatorname{im}(p \circ a) \to \operatorname{Sym}^2 E_C \to \operatorname{coker}(p \circ a) \to 0$$

Before we continue, we calculate the following:

Lemma 4.41. *We have* $\ker(p \circ a) = 0$, *i.e.*, $0 \to F \to \operatorname{Sym}^2 E_C$.

Proof. Assume that $\ker(p \circ a)$ is a line bundle. Then $\operatorname{rk}(\operatorname{im}(p \circ a)) = \operatorname{rk}(F) - 1$ and $\deg(\operatorname{im}(p \circ c)) \geqslant \deg(F) = \mu(F)\operatorname{rk}(F) \geqslant (9 + \frac{1}{7})\operatorname{rk}(f)$. This implies, since $\operatorname{rk}(F) \leqslant 6$,

$$\mu(\operatorname{im}(p \circ a)) \geqslant \frac{(9 + \frac{1}{7})\operatorname{rk}(F)}{\operatorname{rk}(F) - 1} \geqslant \frac{6}{5} \cdot \left(9 + \frac{1}{7}\right) = 384/35 > \mu(\operatorname{Sym}^2 E_C)$$

so we get a contradiction to the polystability of $\operatorname{Sym}^2 E_C$. ∎

First we exclude the case of a destabilizing sub line bundle F. We actually prove that there is no sub line bundle of $\mathcal{N}$ of degree at least five.

Proposition 4.42. *The maximal degree of a sub line bundle of* $\mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$ *is* 4. *In particular, there is no destabilizing sub line bundle.*

Proof. To show the existence of a sub line bundle of degree 4 we choose any $L \in W^4_{12}(C)$ such that $I_2(L) \neq \emptyset$. For instance if the map induced by L is a birational map to a curve in $\mathbb{P}^4$, then the image lies on a quadric. It follows that the Serre dual of L is contained in $\mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$.

Now let A be a line bundle of degree $\geqslant 5$. We will show $H^0(C, \mathcal{N} \otimes A^{-1}) = 0$. The Wahl map of the Serre adjoint $L = K_C \otimes A^{-1}$ is injective by Theorem 4.39. Observe that $H^0(C, \mathcal{N} \otimes A^{-1})$ decomposes as the kernel of this Wahl map and the space of quadrics $I_2(L)$. Since $\mathcal{N}_{C/\mathbb{P}^8}(-1)$ is globally generated by Lemma 4.40, its quotient L has to be as well. We now have to check that $I_2(L) = \emptyset$. In other words, we want the map $\mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$ to be injective for all globally generated line bundles of degree at most 11.

To show this we use that C is general, i.e., it is not a cover of a plane quartic, it has no $\mathfrak{g}^1_5$ and is not a double cover of a plane quintic. We exlude all possibilities by a case by case analysis:

- all line bundles L with $h^0(C, L) = 2$ by the base point free pencil trick.
- $\mathfrak{g}^2_8$ since C is not tetragonal and not a double cover of a plane quartic.
- $\mathfrak{g}^2_9$ since C is not a triple cover of an elliptic curve.
- $\mathfrak{g}^2_{10}$ since there is no $\mathfrak{g}^1_5$ and C is not a double cover of a plane quintic.
- $\mathfrak{g}^3_{10}$ since C is not a double cover of a genus 2 curve, and any quadric containing the image of a birational $\mathfrak{g}^3_{10}$ would induce a $\mathfrak{g}^1_5$.
- $\mathfrak{g}^2_{11}$ since a degree 11 plane curve is not contained in a quadric.
- $\mathfrak{g}^3_{11}$ since any quadric containing the image would induce a $\mathfrak{g}^1_5$. ∎

This has the following important consequence:

Lemma 4.43. *Let* $r = \mathrm{rk}(F)$. *We have* $\mu(F) \leqslant 4 + 9(1 - 1/r)$.

Proof. Let L be a maximal sub line bundle of F. By the main theorem in [MS85] we have
$$\frac{\deg(F) - \deg(L)}{r-1} - \deg(L) \leqslant 9$$
and hence $\mu(F) \leqslant \deg(L) + 9 \cdot (1 - 1/r)$. Since $F \hookrightarrow \mathcal{N}$ we have $\deg(L) \leqslant 4$. ∎

Lemma 4.44. *There is no destabilizing rank 2 subbundle F of $\mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$. More precisely, the second stability degree $s_2(\mathcal{N})$ is bounded below by $s_2(\mathcal{N}) \geqslant 9$.*

Proof. By Lemma 4.43 a maximal rank 2 subbundle F must have $\deg(F) \leqslant 17$. But a destabilizing rank 2 subbundle would need to have $\deg(F) \geqslant 19$. ∎

Our method currently does not yield similar results for subbundles of ranks 3, 4 or 5. However we are able to obtain some bounds.

Lemma 4.45. *If* F *is a destabilizing subbundle of* $\mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$ *then its degree has to be in the following range:*

$$\deg(F) \in \begin{cases} [28,30], & \mathrm{rk}(F) = 3 \\ [37,42], & \mathrm{rk}(F) = 4 \\ [46,52], & \mathrm{rk}(F) = 5 \\ \{56\}, & \mathrm{rk}(F) = 6 \end{cases}$$

Proof. The lower bounds follow from the assumption that F is destabilizing. For rank 3 bundles the upper bound follows from Lemma 4.43. In the rank 4 and 5 case the upper bound follows from the inclusion $F \hookrightarrow \mathrm{Sym}^2 E_C$ and the polystability of $\mathrm{Sym}^2 E_C$. In the rank 5 observe that, additionally, the quotient Q in

$$0 \to F \to \mathrm{Sym}^2 E_C \to Q \to 0$$

needs to have $\deg(Q) \geqslant 12$ by Lemma 4.37, hence $\deg(F) \leqslant 52$. The rank 6 case is treated below. ∎

In the case of $\mathrm{rk}(F) = 6$, the quotient M is a line bundle and we can almost exclude all destabilizing possibilities, except for $M \in W^2_8(C)$.

To do this we will introduce a vector bundle Q, which has many incarnations and will be very useful to us. First we take the second symmetric power of the exact sequence

$$0 \to E_C^\vee \to H^0(C, E_C) \to E_C \to 0$$

to obtain

$$0 \to \wedge^2 E_C^\vee \to H^0(C, E_C) \otimes E_C^\vee \to \mathrm{Sym}^2 H^0(C, E_C) \to \mathrm{Sym}^2 E_C \to 0 \tag{4.9}$$

Now let $Q^\vee$ be the bundle defined by

$$0 \to \wedge^2 E_C^\vee \to H^0(C, E_C) \otimes E_C^\vee \to Q^\vee \to 0 \tag{4.10}$$

so $Q^\vee$ replaces the first two terms in (4.9) and turns out to be the kernel bundle of $\mathrm{Sym}^2 E_C$. Dualizing (4.10), we get

$$0 \to Q \to H^0(C, E_C) \otimes E_C \to \wedge^2 E_C \to 0 \tag{4.11}$$

From this we see that $h^0(C, Q) = 22$ and in fact

$$H^0(C, Q) = \mathrm{Sym}^2 H^0(C, E_C) \oplus \ker(\wedge^2 H^0(C, E_C) \to H^0(C, \wedge^2 E_C))$$

We also see that $H^0(C, Q \otimes A^{-1}) = 0$ for all line bundles A of degree $\geqslant 3$.

In fact, $Q^\vee$ is also the kernel bundle of $\mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$. This follows from the following commutative exact diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & H^0(\mathcal{O}_C) \arrow[r] \arrow[d] & \mathcal{O}_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q^\vee \arrow[r] \arrow[d] & H^0(\mathcal{N}^\vee_{C/\mathbb{P}^8}(2)) \arrow[r] \arrow[d] & \mathcal{N}^\vee_{C/\mathbb{P}^8}(2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q^\vee \arrow[r] \arrow[d] & H^0(\operatorname{Sym}^2 E_C) \arrow[r] \arrow[d] & \operatorname{Sym}^2 E_C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Starting from this fact, we can dualize the kernel bundle sequence and get

$$0 \to \mathcal{N}_{C/\mathbb{P}^8}(-2) \to H^0(C, Q) \to Q \to 0 \tag{4.12}$$

We will use this sequence to show that $\mathcal{N}_{C/\mathbb{P}^8}(-1)$ has no subbundles A of degree $d \geqslant 7$, except possibly for a $\mathfrak{g}^2_8$. Starting from such a bundle A, let $M = \omega_C \otimes A^{-1}$ be the Serre adjoint. Then tensor (4.12) by M and take global sections to obtain

$$0 \to H^0(C, \mathcal{N}_{C/\mathbb{P}^8}(-1) \otimes A^{-1}) \to H^0(C, Q) \otimes H^0(C, M) \to H^0(C, Q \otimes M) \to 0 \tag{4.13}$$

So $K = H^0(C, \mathcal{N}_{C/\mathbb{P}^8}(-1) \otimes A^{-1})$ is the kernel of the multiplication map

$$H^0(C, Q) \otimes H^0(C, M) \to H^0(C, Q \otimes M)$$

Obviously $K = 0$ if $h^0(C, M) \leqslant 1$. If $h^0(C, M) = 2$, then we necessarily have $\deg(M) \geqslant 6$. We can also reinterpret (4.13) in terms of the base point free pencil trick exact sequence of M, tensored with Q:

$$0 \to M^{-1} \otimes Q \to H^0(C, M) \otimes Q \to M \otimes Q \to 0$$

This identifies K with $H^0(C, Q \otimes M^{-1})$, which is zero if $\deg(M) \geqslant 3$.

The last remaining case is $M \in W^2_9(C)$. Because M is a quotient of $\mathcal{N}^\vee_{C/\mathbb{P}^8}(2)$, it is base point free. Just as before, we use the kernel bundle sequence

$$0 \to Q^\vee_M \otimes Q \to H^0(C, M) \otimes Q \to M \otimes Q \to 0$$

to reinterpret K as $H^0(C, Q \otimes Q^\vee_M) = \operatorname{Hom}(Q_M, Q)$. But from (4.11) we have an inclusion $0 \to Q \to H^0(C, E_C) \otimes E_C$ and hence every morphism $Q_M \to Q$ induces a morphism $Q_M \to E_C$. By stability reasons, $\operatorname{Hom}(Q_M, E_C) = 0$, which implies $\operatorname{Hom}(Q_M, Q) = 0$ as well.

4.2.6 An answer to a question by Ciliberto and Miranda

In [CM90], C. Ciliberto and R. Miranda considered and proved the following situation. The elements in the cokernel of the Wahl map

$$\psi_{K_C}\colon \textstyle\bigwedge^2 H^0(C, \omega_C) \to H^0(C, \omega_C^{\otimes 3})$$

on a curve C come from the inclusions of C in K3 surfaces. In genus 9, the general curve is contained in a K3 surface. The dimension of the moduli space of K3 surfaces of genus 9 is $19 + 9 = 28$ while $\dim \mathcal{M}_9 = 24$, so there is a 4-dimensional projective space of K3 surfaces that contain a general genus 9 curve.

On the other hand, $\dim \bigwedge^2 H^0(C, \omega_C) = 36$ and $h^0(C, \omega_C^{\otimes 3}) = 40$. By the previous calculation, the cokernel is a 5-dimensional linear space, hence the kernel of the Gauss map is 1-dimensional. We obtain a line bundle with fiber $\ker(\psi_{K_C})$ over the an open subset of $\mathcal{M}_9$, and the authors ask what this line bundle is.

In this section we calculate the class of the line bundle on the partial compactification of $\mathcal{M}_9$ including only the irreducible nodal curves.

Outline of the calculation

Recall that on every curve we have the sequence

$$0 \to \mathcal{N}^\vee_{C/\mathbb{P}^8}(2) \to \Omega_{\mathbb{P}^8}|_C(2) \to K_C^{\otimes 3} \to 0$$

where

$$\Omega_{\mathbb{P}^8}|_C(2) = M_{K_C} \otimes K_C$$

We globalize this construction. Let $\omega = \omega_\pi$ be the relative dualizing sheaf on the universal curve over $\mathcal{M}_9$. Then we get

$$0 \to \mathcal{N}^\vee \otimes \omega^{\otimes 2} \to \mathcal{M}_\omega \otimes \omega \to \omega^{\otimes 3} \to 0$$

with a bundle $\mathcal{N}$ which restricts to $\mathcal{N}_{C/\mathbb{P}^8}$ on the fibers.

The next ingredient is the globalization of $I_2(K_C)$, the space of quadrics containing the canonical curve. We get this as the kernel of the morphism

$$\mathrm{Sym}^2 \pi_*(\omega) \to \pi_*(\omega^{\otimes 2})$$

of vector bundles on $\mathcal{M}_9$. Now the line bundle we are looking for is the quotient

$$\pi_*\big(\mathcal{N}^\vee \otimes \omega^{\otimes 2}\big)/\mathcal{I}_2(\omega)$$

and we will proceed to calculate its class in various steps.

Chern classes of the global normal bundle

We fix the following notation. Let $\pi\colon \mathcal{C}_g \to \mathcal{M}_g$ be the universal curve and $\omega = \omega_\pi$ be the relative dualizing sheaf. Let $K = c_1(\omega_\pi)$ be its class on $\mathcal{C}_g$ and $\mathbb{E} = \pi_*(\omega_\pi)$ be the Hodge bundle with Hodge classes $\lambda = c_1(\mathbb{E})$ and $\lambda_2 = c_2(\mathbb{E})$. As usual, we also let $\kappa = \pi_*(K^2)$.

Let us start by calculating $c_1\pi_*(\mathcal{M}_\omega \otimes \omega)$. Since $\mathcal{M}_\omega \otimes \omega$ is the kernel of

$$\pi^*(\mathbb{E}) \otimes \omega_\pi \to \omega_\pi^{\otimes 2} \to 0$$

we can pushforward and use the projection formula to obtain

$$0 \to \pi_*(\mathcal{M}_\omega \otimes \omega) \to \mathbb{E} \otimes \mathbb{E} \to \pi_*(\omega_\pi^{\otimes 2}) \to 0 \tag{4.14}$$

Observe that the sequence is exact on the right because it is so on every fiber.

Lemma 4.46. $c_1\pi_*(\omega_\pi^{\otimes 2}) = \lambda + \kappa = 13\lambda - \delta$

Proof. Since $R^1\pi_*(\omega_\pi^{\otimes 2}) = 0$, this is a direct calculation using Grothendieck–Riemann–Roch:

$$\begin{aligned} c_1\pi_*(\omega_\pi^{\otimes 2}) &= \left[\pi_*\left((1 + 2K + 2K^2)\left(1 - \frac{K}{2} + \frac{K + c_2(\mathcal{C}_g)}{12}\right)\right)\right]_1 \\ &= \pi_*\left(\frac{K + c_2(\mathcal{C}_g)}{12} + 2K^2 - K^2\right) \\ &= \lambda + \kappa \end{aligned}$$

■

Corollary 4.47. $c_1\pi_*(\mathcal{M}_\omega \otimes \omega) = (2g-1)\lambda - \kappa$

Proof. This follows from (4.14) and $c_1(\mathbb{E} \otimes \mathbb{E}) = 2gc_1(\mathbb{E}) = 2g\lambda$. ■

The next step is to calculate the first Chern class of $\pi_!\mathcal{F}$ where we define $\mathcal{F} = \mathcal{N}^\vee \otimes \omega_\pi^{\otimes 2}$. We require several preliminary calculations, the first being the Chern classes of $\mathcal{M}_\omega \otimes \omega$ on $\mathcal{C}_g$:

Lemma 4.48. $c_1(\mathcal{M}_\omega) = \lambda - K$, *hence* $c_1(\mathcal{M}_\omega \otimes \omega) = \lambda + (g-2)K$.

Proof. From the exact sequence

$$0 \to \mathcal{M}_\omega \to \pi^*(\mathbb{E}) \to \omega_\pi \to 0$$

we get $c_1(\mathcal{M}_\omega) = \pi^*(\lambda) - K$. The computation of $c_1(\mathcal{M}_\omega \otimes \omega)$ follows from $\mathrm{rk}(\mathcal{M}_\omega) = g - 1$. ■

Lemma 4.49. $c_2(\mathcal{M}_\omega) = \pi^*\lambda_2 + K^2 - \lambda K$, *hence*

$$c_2(\mathcal{M}_\omega \otimes \omega) = \pi^*\lambda_2 + (g-3)\lambda \cdot K + \left(\tbinom{g-2}{2} + 1\right)K^2$$

Proof. Again from the exact sequence $0 \to \mathcal{M}_\omega \to \pi^*(\mathbb{E}) \to \omega_\pi \to 0$ we get

$$\begin{aligned} c_2(\mathcal{M}_\omega) &= c_2\pi^*(\mathbb{E}) + c_1^2(\omega_\pi) - c_1(\pi^*(\mathbb{E})) \cdot c_1(\omega_\pi) \\ &= \pi^*\lambda_2 + K^2 - \lambda \cdot K \end{aligned}$$

For a rank r vector bundle F and a line bundle L we have the general formula

$$c_2(F \otimes L) = c_2(F) + (r-1)c_1(F)c_1(L) + \binom{r}{2} c_1^2(L)$$

from which the second statement follows. ∎

Now we can proceed with the Chern classes of $\mathcal{F}$.

Lemma 4.50. *We have*

$$\begin{aligned} c_1(\mathcal{F}) &= \lambda + (g-5)K \\ c_2(\mathcal{F}) &= \pi^*(\lambda_2) + (g-6)\lambda \cdot K + \left(\tbinom{g-5}{2} + 4\right)K^2 \end{aligned}$$

Proof. This follows essentially from the exact sequence

$$0 \to \mathcal{F} \to \mathcal{M}_\omega \otimes \omega \to \omega_\pi^{\otimes 3} \to 0$$

For brevity, let $M = \mathcal{M}_\omega \otimes \omega$. For the Chern polynomials we get

$$\begin{aligned} c_t(\mathcal{F}) &= c_t(M) \cdot c_t(\omega_\pi^{\otimes 3})^{-1} \\ &= (1 + c_1(M) + c_2(M) + \cdots)(1 - c_1(\omega_\pi^{\otimes 3}) + c_1^2(\omega_\pi^{\otimes 3}) + \cdots) \\ &= (1 + c_1(M) + c_2(M) + \cdots)(1 - 3K + 9K^2 + \cdots) \end{aligned}$$

hence

$$c_1(\mathcal{F}) = c_1(M) - 3K = \lambda + (g-5)K$$

and

$$\begin{aligned} c_2(\mathcal{F}) &= c_2(M) + 9K^2 - 3K \cdot c_1(M) \\ &= \pi^*(\lambda_2) + (g-3)\lambda \cdot K + \left(\tbinom{g-2}{2} + 1\right)K^2 + 9K^2 - 3K(\lambda + (g-2)K) \\ &= \pi^*(\lambda_2) + (g-6)\lambda \cdot K + \left(\tbinom{g-5}{2} + 4\right)K^2 \end{aligned}$$ ∎

We can now apply Grothendieck–Riemann–Roch to $\mathcal{F}$ and π.

Lemma 4.51. *The first Chern class of $\pi_!(\mathcal{F})$ has the following expression:*

$$c_1\pi_*(\mathcal{F}) - c_1R^1\pi_*(\mathcal{F}) = (4g^2 - 26g + 21)\lambda + (g^2 - 11g + 34)\kappa$$

Proof. By Grothendieck–Riemann–Roch we have

$$\begin{aligned}
c_1\pi_!(\mathcal{F}) &= \pi_*\left[\mathrm{ch}(\mathcal{F})\cdot \mathrm{Td}(\pi)\right]_2\\
&= \pi_*\left[\left(g-2+c_1(\mathcal{F})+\frac{1}{2}c_1^2(\mathcal{F})-c_2(\mathcal{F})\right)\left(1-\frac{1}{2}K+\frac{K^2+c_2(\mathcal{C}_g)}{12}\right)\right]_2\\
&= \pi_*\left((g-2)\frac{K^2+c_2(\mathcal{C}_g)}{12}-\frac{1}{2}c_1(\mathcal{F})\cdot K+\frac{1}{2}c_1^2(\mathcal{F})-c_2(\mathcal{F})\right)
\end{aligned}$$

Now we can apply the push-pull formula and get

$$\begin{aligned}
\pi_*(c_1(\mathcal{F})\cdot K) &= \pi_*((\pi^*(\lambda)+(g-5)K)\cdot K)\\
&= (2g-2)\lambda+(g-5)\kappa
\end{aligned}$$

as well as

$$\pi_*(c_1^2(\mathcal{F})) = \lambda\cdot\pi_*\pi^*(\lambda)+2(g-5)(2g-2)\lambda+(g-5)^2\kappa$$

and

$$\pi_*(c_2(\mathcal{F})) = \pi_*\pi^*(\lambda_2)+(2g-2)(g-6)\lambda+\left(\binom{g-5}{2}+4\right)\kappa$$

Observe that $\pi_*\pi^*\alpha = 0$ for any class, since π is not finite. Putting everything together we obtain

$$c_1\pi_!(\mathcal{F}) = (4g^2-26g+21)\lambda+(g^2-11g+34)\kappa \qquad \blacksquare$$

Now observe that

$$c_1R^1\pi_*\mathcal{F} = \pi_*(\mathcal{F}^\vee\otimes\omega_\pi)$$

by Grothendieck duality. From [CM90] we import the exact sequence

$$0\to H^0(C,\omega_C)^\vee\to H^0(C,\mathcal{N}_C(-1))\to[\mathrm{coker}(\psi_{K_C})]^\vee\to 0$$

which we also globalize in the following way:

$$0\to\mathbb{E}^\vee\to\pi_*(\mathcal{F}^\vee\otimes\omega_\pi)\to\mathrm{coker}(\widetilde{\Gamma})^\vee\to 0$$

Here $\widetilde{\Gamma}$ is the pushforward of the map Γ in the exact sequence

$$0\to\mathcal{F}\to\mathcal{M}_\omega\otimes\omega_\pi\xrightarrow{\Gamma}\omega_\pi^{\otimes 3}\to 0$$

i.e., after pushing forward by π, the cokernel of $\widetilde{\Gamma}$ sits in the sequence

$$0\to\pi_*(\mathcal{F})\to\pi_*(\mathcal{M}_\omega\otimes\omega_\pi)\xrightarrow{\widetilde{\Gamma}}\pi_*(\omega_\pi^{\otimes 3})\to\mathrm{coker}(\widetilde{\Gamma})\to 0$$

From these exact sequences we can calculate

$$c_1(\pi_*(\mathcal{F}^\vee\otimes\omega_\pi)) = -\lambda-c_1(\mathrm{coker}\,\widetilde{\Gamma})$$

where the cokernel has first Chern class

$$c_1(\operatorname{coker}\widetilde{\Gamma}) = c_1\pi_*(\omega_\pi^{\otimes 3}) - c_1(\pi_*(\mathcal{M}_\omega \otimes \omega_\pi)) + c_1(\pi_*\mathcal{F})$$

Put together, this gives us

$$c_1(\pi_*(\mathcal{F}^\vee \otimes \omega_\pi)) = -\lambda - c_1(\pi_*\mathcal{F}) - c_1(\pi_*(\omega_\pi^{\otimes 3})) + c_1(\pi_*(\mathcal{M}_\omega \otimes \omega_\pi))$$

Denoting the explicit expression of Lemma 4.51 by $(\star)$ we have

$$\begin{aligned}(\star) &= c_1(\pi_*\mathcal{F}) - c_1(\pi_*(\mathcal{F}^\vee \otimes \omega)) \\ &= 2c_1(\pi_*\mathcal{F}) + \lambda + c_1(\pi_*(\omega_\pi^{\otimes 3})) - c_1(\pi_*(\mathcal{M}_\omega \otimes \omega_\pi))\end{aligned}$$

Just as before, we obtain $c_1(\pi_*(\omega_\pi^{\otimes 3}))$ by an easy GRR calculation:

Lemma 4.52. $c_1(\pi_*(\omega_\pi^{\otimes 3})) = \lambda + 3\kappa$.

Plugging these expressions in, we get

$$c_1(\pi_*(\mathcal{F})) = \frac{1}{2}\big((\star) + (2g-3)\lambda - 4\kappa\big)$$

Chern classes of the globalized ideal of quadrics, and the final result

The next step in the calculation is a globalized $I_2(K_C)$. The map

$$\operatorname{Sym}^2 \pi_*(\omega_\pi) \to \pi_*(\omega_\pi^{\otimes 2})$$

of vector bundles is surjective (since it is surjective on all fibers). Taking the kernel we obtain

$$0 \to \mathcal{I}_2(\omega) \to \operatorname{Sym}^2 \mathbb{E} \to \pi_*(\omega_\pi^{\otimes 2}) \to 0$$

and therefore

$$\begin{aligned}c_1(\mathcal{I}_2(\omega)) &= c_1(\operatorname{Sym}^2 \mathbb{E}) - c_1(\pi_*(\omega_\pi^{\otimes 2})) \\ &= (g+1)c_1(\mathbb{E}) - (\lambda + \kappa) \\ &= g\lambda - \kappa\end{aligned}$$

Finally,

$$c_1(\ker(\psi)) = c_1(\pi_*\mathcal{F}) - c_1(\mathcal{I}_2(\omega))$$

Plugging in the results we previously obtained, and setting $g = 9$, we arrive at

$$c_1\pi_!\mathcal{F} = 111\lambda + 16\kappa$$

and from there

$$c_1\pi_*\mathcal{F} = \frac{1}{2}(111\lambda + 16\kappa + 15\lambda - 4\kappa) = 63\lambda + 6\kappa$$

which yields

$$\begin{aligned}c_1(\ker(\psi)) &= (63\lambda + 6\kappa) - (9\lambda - \kappa) \\ &= 54\lambda + 7\kappa \\ &= 138\lambda - 7\delta\end{aligned}$$

4.3 Instability of the normal bundle in genus 6

Every genus 6 curve is tetragonal, hence by [AFO16, Proposition 3.2], we know that the normal bundles of canonical genus 6 curves are never semistable. The proof in loc. cit. uses a rational scroll induced by the tetragonal pencil. In the spirit of the previous sections, we are going to give an alternative proof for the general curve using Mukai's description of genus 6 curves as quadric sections of Grassmannians.

By [Muk93, Section 5], a canonical genus 6 curve C is an intersection of $G = G(2,5) \subseteq \mathbb{P}^9$ with a 4-dimensional quadric if and only if $W^1_4(C)$ is finite, i.e., if C is not trigonal, bielliptic or a plane quintic. The intersection of G with a $\mathbb{P}^5 = H \subseteq \mathbb{P}^9$ is a del Pezzo surface X and then C is a quadric hypersurface section of S.

The inclusion $C \hookrightarrow G$ in the Grassmannian naturally leads to a destabilizing sequence for the normal bundle $\mathcal{N}_{C/\mathbb{P}^5}$. Let S and Q be the tautological bundle and the universal quotient bundle of G, respectively. We then have (as in section 4.1.1)

$$\mathcal{N}_{X/\mathbb{P}^5} \cong \mathcal{N}_{G(2,5)/\mathbb{P}^9}|_X \cong (\wedge^2 S^\vee) \otimes \det Q$$

Furthermore, the exact sequence of normal bundles

$$0 \to \mathcal{N}_{C/X} \to \mathcal{N}_{C/\mathbb{P}^5} \to \mathcal{N}_{X/\mathbb{P}^5}|_C \to 0$$

is split, i.e.,

$$\mathcal{N}_{C/\mathbb{P}^5} \cong \mathcal{N}_{C/X} \oplus \mathcal{N}_{X/\mathbb{P}^5}|_C = \mathcal{N}_{C/X} \oplus \Big(\bigwedge^2 S_C^\vee\Big)(1)$$

Here S_C denotes the restriction of S to C. Since $S_C^\vee$ is of rank 3 we have

$$\bigwedge^2 S_C^\vee \cong S_C \otimes \det(S_C^\vee) \cong S_C \otimes \omega_C$$

implying $\mathcal{N}_{X/\mathbb{P}^5}|_C = S_C \otimes \omega_C^{\otimes 2}$. Now this is a vector bundle of slope $50/3$ and it is a direct summand of $\mathcal{N}_{C/\mathbb{P}^5}$, which has slope $(\deg \omega_C^{\otimes 7})/4 = 70/4$. Since $50/3 < 70/4$, the normal bundle is unstable.

4.4 Towards the higher genus case

In this section we present some preliminary results and some tools which might prove useful in tackling the question of stability of the normal bundle of a general canonical curve for arbitrary g. The first thing to note is that the results about global generation of $\mathcal{N}_{C/\mathbb{P}^{g-1}}(-1)$ and $\mathcal{N}^\vee_{C/\mathbb{P}^{g-1}}(2)$ are by no means particular to low genus cases:

Lemma 4.53. $\mathcal{N}^\vee_{C/\mathbb{P}^{g-1}}(2)$ *is globally generated for every canonical curve with Clifford index* $\mathrm{Cliff}(C) \geqslant 2$.

Proof. The proof of Lemma 4.38 works in this case as well, since $\mathrm{Cliff}(C) \geqslant 2$ means that C is scheme-theoretically cut out by quadrics. ∎

Lemma 4.54. $\mathcal{N}_{C/\mathbb{P}^{g-1}}(-1)$ *is globally generated for all canonical curves of every genus* g.

Proof. We can copy the proof of Lemma 4.40 word for word. ∎

This has an interesting consequence.

Lemma 4.55. *For every* $r_0 > 0$ *there is an integer* g_0 *such that the normal bundle* $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ *of a general canonical curve of every genus* $g \geqslant g_0$ *has no destabilizing quotient bundle of rank* $r \leqslant r_0$.

Proof. Let $\mathcal{N}_{C/\mathbb{P}^{g-1}}(-1) \to F \to 0$ be a destabilizing quotient of rank r. Then F is globally generated and has slope

$$\mu(F) \leqslant \mu\big(\mathcal{N}_{C/\mathbb{P}^{g-1}}(-1)\big) = 6\left(\frac{g-1}{g-2}\right)$$

We may assume $g \geqslant 8$. Then in particular, $d = \deg(F) \leqslant 7r$. By Lemma 4.8 we must have $h^0(C, \det(F)) \geqslant r+1$. But if g is large enough, then $\rho(g,r,d) < 0$. We conclude that for these g no such F exists. ∎

Remark 4.56. To find the minimal g_0 for a given rank r_0 which this method allows, we have to solve the system

$$d = \left\lfloor 6r\left(\frac{g-1}{g-2}\right)\right\rfloor$$
$$g < (r+1)(g-d+r)$$

and find the minimum g for which both statements hold. As an example, we obtain the following bounds:

r_0	g_0
1	11
2	16
3	21
4	27
5	32

In fact the sequence will continue in the form $g_0 = 5r_0 + 7$.

Note however that this result is purely numerical and does not use the fact that we are considering the normal bundle. In particular, even though the strategy can likely be improved, we cannot expect to obtain stability of $\mathcal{N}_{C/\mathbb{P}^{g-1}}$ in this way.

Another avenue which can possibly be exploited is the connection between cohomology of the normal bundle and surjectivity of certain Wahl maps. Consider a sub line bundle A of $\mathcal{N}_{C/\mathbb{P}^8}(-1)$. The existence of A then implies that the Wahl map $\psi_{K_C,K_C\otimes A}$ cannot be surjective.

Lemma 4.57. *A line bundle A of degree $\deg(A) \geqslant 3$ is a sub line bundle of $\mathcal{N}_{C/\mathbb{P}^8}(-1)$ if and only if the Wahl map $\psi_{K_C,K_C\otimes A}$ is not surjective.*

Proof. By Serre duality we have

$$H^0(C, \mathcal{N}_{C/\mathbb{P}^8}(-1) \otimes A^{-1}) \cong H^1(C, \mathcal{N}^{\vee}_{C/\mathbb{P}^8}(2) \otimes A)^{\vee}$$

Consider the twisted normal bundle exact sequence

$$0 \to \mathcal{N}^{\vee}_{C/\mathbb{P}^8}(2) \otimes A \to M_{K_C} \otimes K_C \otimes A \to K_C^{\otimes 3} \otimes A \to 0$$

The stability of M_{K_C} and the assumption on $\deg(A)$ together imply that $H^1(C, M_{K_C} \otimes K_C \otimes A) = 0$. Then the long exact sequence in cohomology shows that $H^1(C, \mathcal{N}^{\vee}_{C/\mathbb{P}^8}(2) \otimes A) = 0$ if and only if the map

$$H^0(C, M_{K_C} \otimes K_C \otimes A) \to H^0(C, K_C^{\otimes 3} \otimes A)$$

is surjective. This map is precisely the Wahl map $\psi_{K_C,K_C\otimes A}$. ∎

Questions about the surjectivity of Wahl maps of this type are studied in detail for instance in [Pao95]. However, in general it seems hard to decide whether $\psi_{K_C,K_C\otimes A}$ has cokernel, even when A is a very special line bundle.

The last route we want to explore is other embeddings of C, giving a factorization of the canonical embedding. We already saw an example of this in section 4.1.3. Consider any $A \in W^r_d(C)$ and its Serre dual $L \in W^{r'}_{d'}$ where $d' = 2g-2-d$ and $r' = g+r-d-1$. If both A and L are globally generated, then the product linear system $|A| \times |L|$ gives a map $\phi_{A,L} \colon C \to \mathbb{P}^r \times \mathbb{P}^{r'}$, which often is an embedding. Composing this with the Segre embedding $\mathbb{P}^r \times \mathbb{P}^{r'}$ yields an embedding $C \to \mathbb{P}^N$ with $\mathcal{O}_C(1) = \omega_C$, where $N = (r+1)(r'+1)-1$. We distinguish several cases:

1. If $\rho(g,r,d) > 0$ then $N < g$, hence C is embedded by a linear subsystem of the canonical embedding and this case is not interesting for our purposes. The subsystem in question is the image of the Petri map of A.

2. If instead $\rho(g,r,d) < 0$ then the general curve has no such $\mathfrak{g}^r_d$, so we cannot study its normal bundle in this way. Furthermore, the embedding $C \to \mathbb{P}^N$ will be degenerate. We can however choose the $\mathbb{P}^{g-1}$ containing C and hope to study the normal bundle of non-general curves this way. In order for this to be possible, the Petri map of A has to be surjective.

3. The most interesting case is $\rho(g, r, d) = 0$. Now a general curve will have a finite number of such A and L, both of them are globally generated, $\varphi_{A,L}$ is an embedding and the composition with the Segre embedding $C \to \mathbb{P}^N$ is precisely the canonical embedding. So we obtain an exact sequence

$$0 \to \mathcal{N}_{C/\mathbb{P}^r \times \mathbb{P}^{r'}} \to \mathcal{N}_{C/\mathbb{P}^{g-1}} \to \mathcal{N}_{\mathbb{P}^r \times \mathbb{P}^{r'}/\mathbb{P}^N}|_C \to 0$$

Using the general form of Lemma 4.18 we get

$$\mathcal{N}_{\mathbb{P}^r \times \mathbb{P}^{r'}/\mathbb{P}^N}|_C(-1) \cong Q_A \otimes Q_L$$

where Q_A and Q_L are the duals of the kernel bundles of A and L, respectively. If $\mathcal{N}_{C/\mathbb{P}^r \times \mathbb{P}^{r'}}$ can be understood more explicitly, the above exact sequence can be used to obtain more stability results about $\mathcal{N}_{C/\mathbb{P}^{g-1}}(-1)$. A particular case to keep in mind is when one of the line bundles, say A, already induces an embedding $C \to \mathbb{P}^r$. Then the normal bundle $\mathcal{N}_{C/\mathbb{P}^r}$ appears in the exact sequence

$$0 \to \varphi_L^* \Omega_{\mathbb{P}^r}^{\vee} \to \mathcal{N}_{C/\mathbb{P}^r \times \mathbb{P}^{r'}} \to \mathcal{N}_{C/\mathbb{P}^r} \to 0$$

Example 4.58. Consider again a general genus 9 curve. Two Serre dual line bundles $\alpha, \beta \in W_8^2(C)$ induce an embedding $\varphi\colon C \to \mathbb{P}^2 \times \mathbb{P}^2$. The Segre embedding $\sigma\colon \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ with image Σ embeds $\varphi(C)$ canonically in $\mathbb{P}^8$. We get an exact sequence

$$0 \to \mathcal{N}_{C/\Sigma} \to \mathcal{N}_{C/\mathbb{P}^8} \to \mathcal{N}_{\Sigma/\mathbb{P}^8}|_C \to 0$$

which, more explicitly, after twisting is

$$0 \to \mathcal{N}_{C/\mathbb{P}^2 \times \mathbb{P}^2}(-1, -1) \to \mathcal{N}_{C/\mathbb{P}^8}(-1) \to Q_\alpha \otimes Q_\beta \to 0$$

Finally, let us note that in general it seems to be hard to decide when precisely the normal bundle of an arbitrary (non-general) curve C is unstable. Not having a tetragonal pencil does not seem to be the right condition:

Example 4.59. The normal bundle of a canonical curve can be unstable even if the curve is not tetragonal. Consider a curve of genus 9 with a $\mathfrak{g}_9^3 = L$ but no $\mathfrak{g}_4^1$. Such curves are explicitly constructed in M. Sagraloff's PhD thesis ([Sag06]). We have $\rho(9, 3, 9) = -3$ as well as $\rho(9, 1, 4) = -3$, but the two Brill–Noether loci $\mathcal{M}_{9,4}^1$ and $\mathcal{M}_{9,9}^3$ are not equal.

Consider the Serre dual $A = \omega_C \otimes L^{-1}$ of L, which is a $\mathfrak{g}_7^2$. By using L and A, we obtain an embedding of C into $\mathbb{P}^2 \times \mathbb{P}^3$. As before, by composing this with the Segre embedding, we recover the canonical image of C. Hence we get a quotient $Q_L \otimes Q_A$ of the normal bundle of slope $\mu(Q_A \otimes Q_L) = 13/2$. However, note that the normal bundle has slope $\mu(\mathcal{N}_{C/\mathbb{P}^8}(-1)) = 48/7$, which is strictly bigger.

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 02. Januar 2017 Gregor Bruns